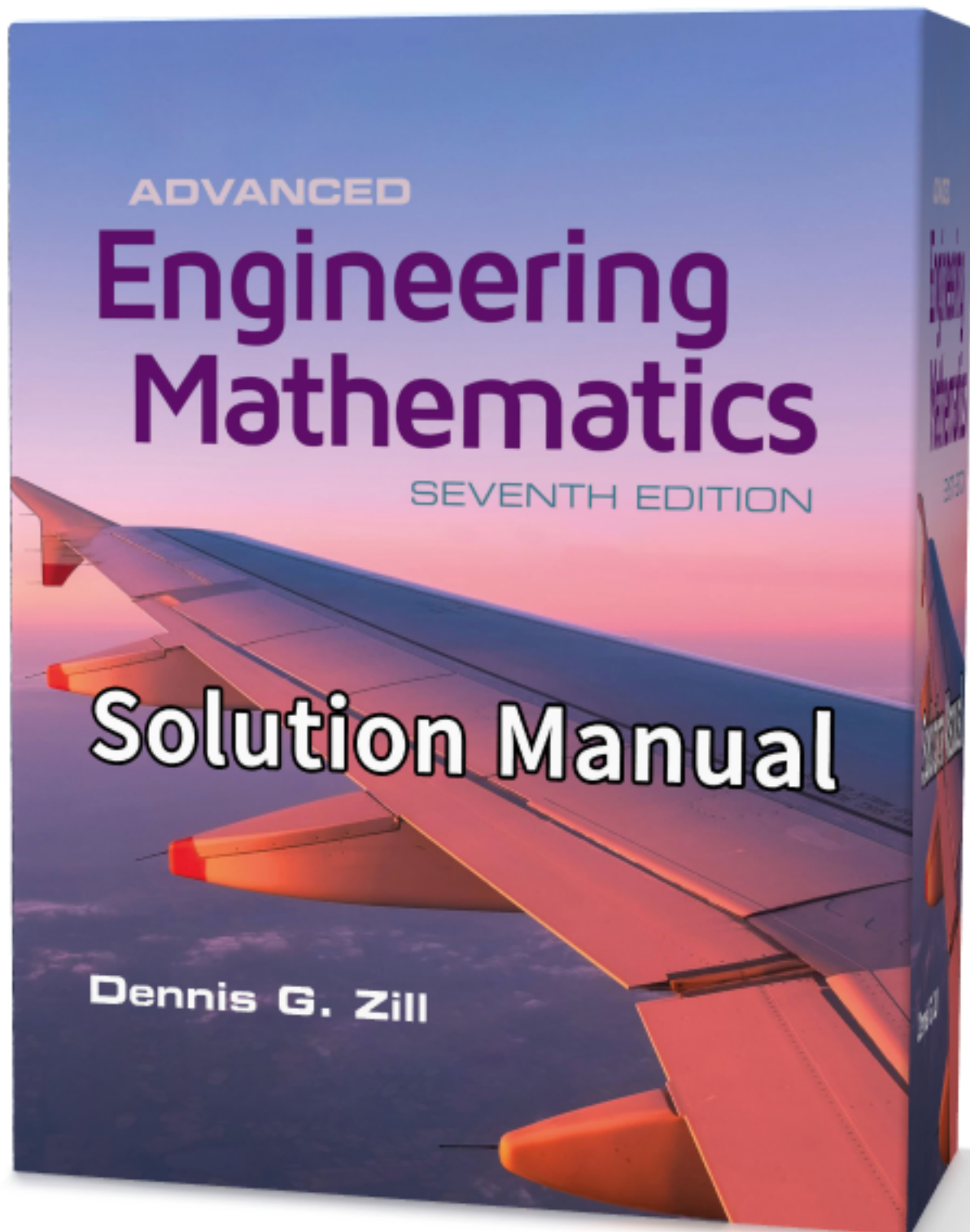




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SOLUTION MANUAL

Chapter 1

Introduction to Differential Equations

1.1 Definitions and Terminology

1. Second order; linear
2. Third order; nonlinear because of $(dy/dx)^4$
3. Fourth order; linear
4. Second order; nonlinear because of $\cos(r + u)$
5. Second order; nonlinear because of $(dy/dx)^2$ or $\sqrt{1 + (dy/dx)^2}$
6. Second order; nonlinear because of R^2
7. Third order; linear
8. Second order; nonlinear because of $\dot{x}^2$
9. First order; nonlinear because of $\sin\left(\frac{dy}{dx}\right)$
10. First order; linear in x .
11. Writing the differential equation in the form $x(dy/dx) + y^2 = 1$, we see that it is nonlinear in y because of y^2 . However, writing it in the form $(y^2 - 1)(dx/dy) + x = 0$, we see that it is linear in x .
12. Writing the differential equation in the form $u(dv/du) + (1 + u)v = ue^u$, we see that it is linear in v . However, writing it in the form $(v + uv - ue^u)(du/dv) + u = 0$, we see that it is nonlinear in u .
13. From $y = e^{-x/2}$ we obtain $y' = -\frac{1}{2}e^{-x/2}$. Then $2y' + y = -e^{-x/2} + e^{-x/2} = 0$.
14. From $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$ we obtain $dy/dt = 24e^{-20t}$, so that

$$\frac{dy}{dt} + 20y = 24e^{-20t} + 20\left(\frac{6}{5} - \frac{6}{5}e^{-20t}\right) = 24.$$

15. From $y = e^{3x} \cos 2x$ we obtain $y' = 3e^{3x} \cos 2x - 2e^{3x} \sin 2x$ and $y'' = 5e^{3x} \cos 2x - 12e^{3x} \sin 2x$, so that $y'' - 6y' + 13y = 0$.

16. From $y = -\cos x \ln(\sec x + \tan x)$ we obtain $y' = -1 + \sin x \ln(\sec x + \tan x)$ and $y'' = \tan x + \cos x \ln(\sec x + \tan x)$. Then $y'' + y = \tan x$.

17. The domain of the function, found by solving $x + 2 \geq 0$, is $[-2, \infty)$. From $y' = 1 + 2(x + 2)^{-1/2}$ we have

$$\begin{aligned}(y - x)y' &= (y - x)[1 + (2(x + 2))^{-1/2}] \\ &= y - x + 2(y - x)(x + 2)^{-1/2} \\ &= y - x + 2[x + 4(x + 2)^{1/2} - x](x + 2)^{-1/2} \\ &= y - x + 8(x + 2)^{1/2}(x + 2)^{-1/2} = y - x + 8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2, \infty)$ because y' is not defined at $x = -2$.

18. Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, n an integer, the domain of $y = 5 \tan 5x$ is $\{x \mid 5x \neq \pi/2 + n\pi\}$ or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25 \sec^2 5x$ we have

$$y' = 25(1 + \tan^2 5x) = 25 + 25 \tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. Another interval is $(\pi/10, 3\pi/10)$, and so on.

19. The domain of the function is $\{x \mid 4 - x^2 \neq 0\}$ or $\{x \mid x \neq -2 \text{ or } x \neq 2\}$. From $y' = 2x/(4 - x^2)^2$ we have

$$y' = 2x \left(\frac{1}{4 - x^2} \right)^2 = 2xy^2.$$

An interval of definition for the solution of the differential equation is $(-2, 2)$. Other intervals are $(-\infty, -2)$ and $(2, \infty)$.

20. The function is $y = 1/\sqrt{1 - \sin x}$, whose domain is obtained from $1 - \sin x \neq 0$ or $\sin x \neq 1$. Thus, the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1 - \sin x)^{-3/2}(-\cos x)$ we have

$$2y' = (1 - \sin x)^{-3/2} \cos x = [(1 - \sin x)^{-1/2}]^3 \cos x = y^3 \cos x.$$

An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. Another one is $(5\pi/2, 9\pi/2)$, and so on.

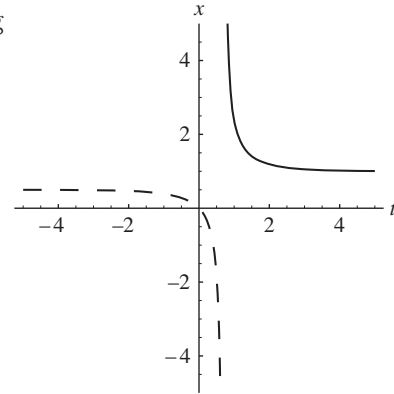
21. Writing $\ln(2X - 1) - \ln(X - 1) = t$ and differentiating implicitly we obtain

$$\frac{2}{2X - 1} \frac{dX}{dt} - \frac{1}{X - 1} \frac{dX}{dt} = 1$$

$$\left(\frac{2}{2X - 1} - \frac{1}{X - 1} \right) \frac{dX}{dt} = 1$$

$$\frac{2X - 2 - 2X + 1}{(2X - 1)(X - 1)} \frac{dX}{dt} = 1$$

$$\frac{dX}{dt} = -(2X - 1)(X - 1) = (X - 1)(1 - 2X).$$



Exponentiating both sides of the implicit solution we obtain

$$\frac{2X - 1}{X - 1} = e^t$$

$$2X - 1 = Xe^t - e^t$$

$$(e^t - 1) = (e^t - 2)X$$

$$X = \frac{e^t - 1}{e^t - 2}.$$

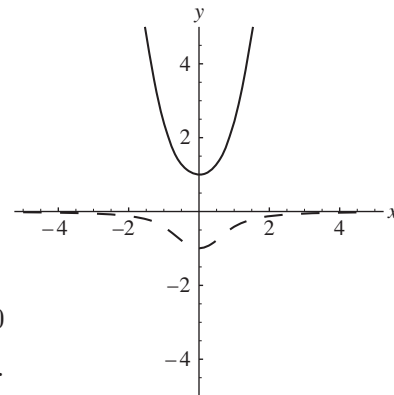
Solving $e^t - 2 = 0$ we get $t = \ln 2$. Thus the solution is defined on $(-\infty, \ln 2)$ or on $(\ln 2, \infty)$. The graph of the solution defined on $(-\infty, \ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2, \infty)$ is solid.

22. Implicitly differentiating the solution, we obtain

$$-2x^2 \frac{dy}{dx} - 4xy + 2y \frac{dy}{dx} = 0$$

$$-x^2 dy - 2xy dx + y dy = 0$$

$$2xy dx + (x^2 - y) dy = 0.$$



Using the quadratic formula to solve $y^2 - 2x^2y - 1 = 0$ for y , we get $y = (2x^2 \pm \sqrt{4x^4 + 4})/2 = x^2 \pm \sqrt{x^4 + 1}$. Thus, two explicit solutions are $y_1 = x^2 + \sqrt{x^4 + 1}$ and $y_2 = x^2 - \sqrt{x^4 + 1}$. Both solutions are defined on $(-\infty, \infty)$. The graph of $y_1(x)$ is solid and the graph of y_2 is dashed.

23. Differentiating $P = c_1 e^t / (1 + c_1 e^t)$ we obtain

$$\begin{aligned} \frac{dP}{dt} &= \frac{(1 + c_1 e^t) c_1 e^t - c_1 e^t \cdot c_1 e^t}{(1 + c_1 e^t)^2} = \frac{c_1 e^t}{1 + c_1 e^t} \frac{[(1 + c_1 e^t) - c_1 e^t]}{1 + c_1 e^t} \\ &= \frac{c_1 e^t}{1 + c_1 e^t} \left[1 - \frac{c_1 e^t}{1 + c_1 e^t} \right] = P(1 - P). \end{aligned}$$

24. Differentiating $y = 2x^2 - 1 + c_1 e^{-2x^2}$ we obtain $\frac{dy}{dx} = 4x - 4xc_1 e^{-2x^2}$, so that

$$\frac{dy}{dx} + 4xy = 4x - 4xc_1 e^{-2x^2} + 8x^3 - 4x + 4c_1 x e^{-x^2} = 8x^3$$

25. From $y = c_1 e^{2x} + c_2 x e^{2x}$ we obtain $\frac{dy}{dx} = (2c_1 + c_2)e^{2x} + 2c_2 x e^{2x}$ and $\frac{d^2 y}{dx^2} = (4c_1 + 4c_2)e^{2x} + 4c_2 x e^{2x}$, so that

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = (4c_1 + 4c_2 - 8c_1 - 4c_2 + 4c_1)e^{2x} + (4c_2 - 8c_2 + 4c_2)x e^{2x} = 0.$$

26. From $y = c_1 x^{-1} + c_2 x + c_3 x \ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1 x^{-2} + c_2 + c_3 + c_3 \ln x + 8x,$$

$$\frac{d^2 y}{dx^2} = 2c_1 x^{-3} + c_3 x^{-1} + 8,$$

and

$$\frac{d^3 y}{dx^3} = -6c_1 x^{-4} - c_3 x^{-2},$$

so that

$$\begin{aligned} x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y &= (-6c_1 + 4c_1 + c_1 + c_1)x^{-1} + (-c_3 + 2c_3 - c_2 - c_3 + c_2)x \\ &\quad + (-c_3 + c_3)x \ln x + (16 - 8 + 4)x^2 \\ &= 12x^2 \end{aligned}$$

In Problems 27–30, we use the Product Rule and the derivative of an integral ((12) of this section):

$$\frac{d}{dx} \int_a^x g(t) dt = g(x).$$

27. Differentiating $y = e^{3x} \int_1^x \frac{e^{-3t}}{t} dt$ we obtain $\frac{dy}{dx} = e^{3x} \int_1^x \frac{e^{-3t}}{t} dt + \frac{e^{-3x}}{x} \cdot e^{3x}$ or

$$\frac{dy}{dx} = e^{3x} \int_1^x \frac{e^{-3t}}{t} dt + \frac{1}{x}, \text{ so that}$$

$$\begin{aligned} x \frac{dy}{dx} - 3xy &= x \left(e^{3x} \int_1^x \frac{e^{-3t}}{t} dt + \frac{1}{x} \right) - 3x \left(e^{3x} \int_1^x \frac{e^{-3t}}{t} dt \right) \\ &= x e^{3x} \int_1^x \frac{e^{-3t}}{t} dt + 1 - 3x e^{3x} \int_1^x \frac{e^{-3t}}{t} dt = 1 \end{aligned}$$

28. Differentiating $y = \sqrt{x} \int_4^x \frac{\cos t}{\sqrt{t}} dt$ we obtain $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \int_4^x \frac{\cos t}{\sqrt{t}} dt + \frac{\cos x}{\sqrt{x}} \cdot \sqrt{x}$ or
 $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \int_4^x \frac{\cos t}{\sqrt{t}} dt + \cos x$, so that

$$\begin{aligned} 2x \frac{dy}{dx} - y &= 2x \left(\frac{1}{2\sqrt{x}} \int_4^x \frac{\cos t}{\sqrt{t}} dt + \cos x \right) - \sqrt{x} \int_4^x \frac{\cos t}{\sqrt{t}} dt \\ &= \sqrt{x} \int_4^x \frac{\cos t}{\sqrt{t}} dt + 2x \cos x - \sqrt{x} \int_4^x \frac{\cos t}{\sqrt{t}} dt = 2x \cos x \end{aligned}$$

29. Differentiating $y = \frac{5}{x} + \frac{10}{x} \int_1^x \frac{\sin t}{t} dt$ we obtain $\frac{dy}{dx} = -\frac{5}{x^2} - \frac{10}{x^2} \int_1^x \frac{\sin t}{t} dt + \frac{\sin x}{x} \cdot \frac{10}{x}$ or
 $\frac{dy}{dx} = -\frac{5}{x^2} - \frac{10}{x^2} \int_1^x \frac{\sin t}{t} dt + \frac{10 \sin x}{x^2}$, so that

$$\begin{aligned} x^2 \frac{dy}{dx} + xy &= x^2 \left(-\frac{5}{x^2} - \frac{10}{x^2} \int_1^x \frac{\sin t}{t} dt + \frac{10 \sin x}{x^2} \right) + x \left(\frac{5}{x} + \frac{10}{x} \int_1^x \frac{\sin t}{t} dt \right) \\ &= -5 - 10 \int_1^x \frac{\sin t}{t} dt + 10 \sin x + 5 + 10 \int_1^x \frac{\sin t}{t} dt = 10 \sin x \end{aligned}$$

30. Differentiating $y = e^{-x^2} + e^{-x^2} \int_0^x e^{t^2} dt$ we obtain $\frac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + e^{x^2} \cdot e^{-x^2}$
 or $\frac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + 1$, so that

$$\begin{aligned} \frac{dy}{dx} + 2xy &= \left(-2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + 1 \right) + 2x \left(e^{-x^2} + e^{-x^2} \int_0^x e^{t^2} dt \right) \\ &= -2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + 1 + 2xe^{-x^2} + 2xe^{-x^2} \int_0^x e^{t^2} dt = 1 \end{aligned}$$

31. From

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

we obtain

$$y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

so that $xy' - 2y = 0$.

32. The function $y(x)$ is not continuous at $x = 0$ since $\lim_{x \rightarrow 0^-} y(x) = 5$ and $\lim_{x \rightarrow 0^+} y(x) = -5$. Thus,
 $y'(x)$ does not exist at $x = 0$.

33. Force the function $y = e^{mx}$ into the equation $y' + 2y = 0$ to get

$$(e^{mx})' + 2(e^{mx}) = 0$$

$$me^{mx} + 2e^{mx} = 0$$

$$e^{mx}(m + 2) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = -2$ and so $y = e^{-2x}$ is a solution.

34. Force the function $y = e^{mx}$ into the equation $3y' - 4y = 0$ to get

$$3(e^{mx})' - 4(e^{mx}) = 0$$

$$3me^{mx} - 4e^{mx} = 0$$

$$e^{mx}(3m - 4) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = 4/3$ and so $y = e^{4x/3}$ is a solution.

35. Force the function $y = e^{mx}$ into the equation $y'' - 5y' + 6y = 0$ to get

$$(e^{mx})'' - 5(e^{mx})' + 6(e^{mx}) = 0$$

$$m^2e^{mx} - 5me^{mx} + 6e^{mx} = 0$$

$$e^{mx}(m^2 - 5m + 6) = 0$$

$$e^{mx}(m - 2)(m - 3) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = 2$ and $m = 3$ therefore $y = e^{2x}$ and $y = e^{3x}$ are solutions.

36. Force the function $y = e^{mx}$ into the equation $2y'' + 9y' - 5y = 0$ to get

$$2(e^{mx})'' + 9(e^{mx})' - 5(e^{mx}) = 0$$

$$2m^2e^{mx} + 9me^{mx} - 5e^{mx} = 0$$

$$e^{mx}(2m^2 + 9m - 5) = 0$$

$$e^{mx}(m + 5)(2m - 1) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = -5$ and $m = 1/2$ therefore $y = e^{-5x}$ and $y = e^{x/2}$ are solutions.

37. Force the function $y = x^m$ into the equation $xy'' + 2y' = 0$ to get

$$x \cdot (x^m)'' + 2(x^m)' = 0$$

$$x \cdot m(m - 1)x^{m-2} + 2mx^{m-1} = 0$$

$$(m^2 - m)x^{m-1} + 2mx^{m-1} = 0$$

$$x^{m-1}[m^2 + m] = 0$$

$$x^{m-1}[m(m + 1)] = 0$$

The last line implies that $m = 0$ and $m = -1$ therefore $y = x^0 = 1$ and $y = x^{-1}$ are solutions.

38. Force the function $y = x^m$ into the equation $4x^2y'' + y = 0$ to get

$$4x^2 \cdot (x^m)'' + (x^m) = 0$$

$$4x^2 \cdot m(m-1)x^{m-2} + x^m = 0$$

$$4(m^2 - m)x^m + x^m = 0$$

$$x^m[4m^2 - 4m + 1] = 0$$

$$x^m[(2m-1)(2m-1)] = 0$$

The last line implies $m = 1/2$ therefore $y = x^{1/2}$ is a solution.

39. Force the function $y = x^m$ into the equation $x^2y'' - 7xy' + 15y = 0$ to get

$$x^2 \cdot (x^m)'' - 7x \cdot (x^m)' + 15(x^m) = 0$$

$$x^2 \cdot m(m-1)x^{m-2} - 7x \cdot mx^{m-1} + 15x^m = 0$$

$$(m^2 - m)x^m - 7mx^m + 15x^m = 0$$

$$x^m[m^2 - 8m + 15] = 0$$

$$x^m[(m-3)(m-5)] = 0$$

The last line implies that $m = 3$ and $m = 5$ therefore $y = x^3$ and $y = x^5$ are solutions.

40. Force the function $y = x^m$ into the equation $x^2y''' - 3xy'' + 3y' = 0$ to get

$$x^2 \cdot (x^m)''' - 3x \cdot (x^m)'' + 3(x^m)' = 0$$

$$x^2 \cdot m(m-1)(m-2)x^{m-3} - 3x \cdot m(m-1)x^{m-2} + 3 \cdot mx^{m-1} = 0$$

$$(m^3 - 3m^2 + 2m)x^{m-1} - 3(m^2 - m)x^{m-1} + 3mx^{m-1} = 0$$

$$x^{m-1}[m^3 - 6m^2 + 8m] = 0$$

$$x^m[m(m-2)(m-4)] = 0$$

The last line implies that $m = 0$, $m = 2$ and $m = 4$ therefore $y = x^0 = 1$, $y = x^2$, and $y = x^4$ are solutions.

In Problems 41–44, we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$.

41. Solving $5c = 10$ we see that $y = 2$ is a constant solution.

42. Solving $c^2 + 2c - 3 = (c+3)(c-1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.

43. Since $1/(c-1) = 0$ has no solutions, the differential equation has no constant solutions.

44. Solving $6c = 10$ we see that $y = 5/3$ is a constant solution.

45. From $y = (x + c_1)^2$ we obtain

$$\left(\frac{dy}{dx}\right)^2 = (2(x + c_1))^2 = 4(x + c_1)^2.$$

Then

$$4y = 4(x + c_1)^2.$$

Inspection of the differential equation reveals that $y = 0$ is a solution of the differential equation but is not a member of the one-parameter family $y = (x + c_1)^2$.

46. From $y = 3 \sin(x + c_1)$ we obtain

$$\left(\frac{dy}{dx}\right)^2 = (3 \cos(x + c_1))^2 = 9 \cos^2(x + c_1).$$

Then

$$9 - y^2 = 9 - (3 \sin(x + c_1))^2 = 9 - 9 \sin^2(x + c_1) = 9(1 - \sin^2(x + c_1)) = 9 \cos^2(x + c_1).$$

Inspection of the differential equation reveals that $y = 3$ and $y = -3$ are solutions of the differential equation but not members of the one-parameter family $y = 3 \sin(x + c_1)$.

47. From $x - \sqrt{16 - y^2} = c_1$ we obtain

$$\begin{aligned} \frac{d}{dx} (x - \sqrt{16 - y^2}) &= \frac{d}{dx} (c_1) \\ 1 - \frac{1}{2} (16 - y^2)^{-1/2} \left(-2y \frac{dy}{dx}\right) &= 0 \\ 1 + \frac{y}{\sqrt{16 - y^2}} \frac{dy}{dx} &= 0 \\ \frac{y}{\sqrt{16 - y^2}} \frac{dy}{dx} &= -1 \\ y \frac{dy}{dx} &= -\sqrt{16 - y^2}. \end{aligned}$$

Then

$$y \frac{dy}{dx} + \sqrt{16 - y^2} = -\sqrt{16 - y^2} + \sqrt{16 - y^2} = 0.$$

Inspection of the differential equation reveals that $y = -4$ and $y = 4$ are solutions of the given differential equation but are not members of the one-parameter family $x - \sqrt{16 - y^2} = c_1$.

48. From $y = x - (x - c_1)^2$ we obtain

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2 &= (1 - 2(x - c_1))^2 = 1 - 4(x - c_1) + 4(x - c_1)^2 = 1 - 4x + 4c_1 + 4x^2 - 8xc_1 + 4c_1^2 \\ -2\frac{dy}{dx} &= -2(1 - 2(x - c_1)) = -2 + 4x - 4c_1 \\ 4y &= 4(x - (x - c_1)^2) = 4(x - x^2 + 2xc_1 - c_1^2) = 4x - 4x^2 + 8xc_1 - 4c_1^2.\end{aligned}$$

Then

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 4y = 4x - 1.$$

Inspection of the differential equation reveals that $y = x$ is a solution of the differential equation but not a member of the one-parameter family $y = x - (x - c_1)^2$.

49. From $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ we obtain

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad \text{and} \quad \frac{dy}{dt} = 2e^{-2t} + 30e^{6t}.$$

Then

$$x + 3y = (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = -2e^{-2t} + 18e^{6t} = \frac{dx}{dt}$$

and

$$5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = 2e^{-2t} + 30e^{6t} = \frac{dy}{dt}.$$

50. From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}.$$

51. $(y')^2 + 1 = 0$ has no real solutions because $(y')^2 + 1$ is positive for all differentiable functions $y = \phi(x)$.

52. The only solution of $(y')^2 + y^2 = 0$ is $y = 0$, since if $y \neq 0$, $y^2 > 0$ and $(y')^2 + y^2 \geq y^2 > 0$.

53. The first derivative of $f(x) = e^x$ is e^x . The first derivative of $f(x) = e^{kx}$ is ke^{kx} . The differential equations are $y' = y$ and $y' = ky$, respectively.
54. Any function of the form $y = ce^x$ or $y = ce^{-x}$ is its own second derivative. The corresponding differential equation is $y'' - y = 0$. Functions of the form $y = c \sin x$ or $y = c \cos x$ have second derivatives that are the negatives of themselves. The differential equation is $y'' + y = 0$.
55. We first note that $\sqrt{1 - y^2} = \sqrt{1 - \sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$. This prompts us to consider values of x for which $\cos x < 0$, such as $x = \pi$. In this case

$$\left. \frac{dy}{dx} \right|_{x=\pi} = \left. \frac{d}{dx}(\sin x) \right|_{x=\pi} = \cos x|_{x=\pi} = \cos \pi = -1,$$

but

$$\sqrt{1 - y^2}|_{x=\pi} = \sqrt{1 - \sin^2 \pi} = \sqrt{1} = 1.$$

Thus, $y = \sin x$ will only be a solution of $y' = \sqrt{1 - y^2}$ when $\cos x > 0$. An interval of definition is then $(-\pi/2, \pi/2)$. Other intervals are $(3\pi/2, 5\pi/2)$, $(7\pi/2, 9\pi/2)$, and so on.

56. Since the first and second derivatives of $\sin t$ and $\cos t$ involve $\sin t$ and $\cos t$, it is plausible that a linear combination of these functions, $A \sin t + B \cos t$, could be a solution of the differential equation. Using $y' = A \cos t - B \sin t$ and $y'' = -A \sin t - B \cos t$ and substituting into the differential equation we get

$$\begin{aligned} y'' + 2y' + 4y &= -A \sin t - B \cos t + 2A \cos t - 2B \sin t + 4A \sin t + 4B \cos t \\ &= (3A - 2B) \sin t + (2A + 3B) \cos t = 5 \sin t \end{aligned}$$

Thus $3A - 2B = 5$ and $2A + 3B = 0$. Solving these simultaneous equations we find $A = \frac{15}{13}$ and $B = -\frac{10}{13}$. A particular solution is $y = \frac{15}{13} \sin t - \frac{10}{13} \cos t$.

57. One solution is given by the upper portion of the graph with domain approximately $(0, 2.6)$. The other solution is given by the lower portion of the graph, also with domain approximately $(0, 2.6)$.
58. One solution, with domain approximately $(-\infty, 1.6)$ is the portion of the graph in the second quadrant together with the lower part of the graph in the first quadrant. A second solution, with domain approximately $(0, 1.6)$ is the upper part of the graph in the first quadrant. The third solution, with domain $(0, \infty)$, is the part of the graph in the fourth quadrant.

59. Differentiating $(x^3 + y^3)/xy = 3c$ we obtain

$$\begin{aligned}\frac{xy(3x^2 + 3y^2y') - (x^3 + y^3)(xy' + y)}{x^2y^2} &= 0 \\ 3x^3y + 3xy^3y' - x^4y' - x^3y - xy^3y' - y^4 &= 0 \\ (3xy^3 - x^4 - xy^3)y' &= -3x^3y + x^3y + y^4 \\ y' &= \frac{y^4 - 2x^3y}{2xy^3 - x^4} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.\end{aligned}$$

60. A tangent line will be vertical where y' is undefined, or in this case, where $x(2y^3 - x^3) = 0$. This gives $x = 0$ and $2y^3 = x^3$. Substituting $y^3 = x^3/2$ into $x^3 + y^3 = 3xy$ we get

$$\begin{aligned}x^3 + \frac{1}{2}x^3 &= 3x \left(\frac{1}{2^{1/3}} x \right) \\ \frac{3}{2}x^3 &= \frac{3}{2^{1/3}}x^2 \\ x^3 &= 2^{2/3}x^2 \\ x^2(x - 2^{2/3}) &= 0.\end{aligned}$$

Thus, there are vertical tangent lines at $x = 0$ and $x = 2^{2/3}$, or at $(0, 0)$ and $(2^{2/3}, 2^{1/3})$. Since $2^{2/3} \approx 1.59$, the estimates of the domains in Problem 58 were close.

61. The derivatives of the functions are $\phi_1'(x) = -x/\sqrt{25 - x^2}$ and $\phi_2'(x) = x/\sqrt{25 - x^2}$, neither of which is defined at $x = \pm 5$.

62. To determine if a solution curve passes through $(0, 3)$ we let $t = 0$ and $P = 3$ in the equation $P = c_1 e^t / (1 + c_1 e^t)$. This gives $3 = c_1 / (1 + c_1)$ or $c_1 = -\frac{3}{2}$. Thus, the solution curve

$$P = \frac{(-3/2)e^t}{1 - (3/2)e^t} = \frac{-3e^t}{2 - 3e^t}$$

passes through the point $(0, 3)$. Similarly, letting $t = 0$ and $P = 1$ in the equation for the one-parameter family of solutions gives $1 = c_1 / (1 + c_1)$ or $c_1 = 1 + c_1$. Since this equation has no solution, no solution curve passes through $(0, 1)$.

63. For the first-order differential equation integrate $f(x)$. For the second-order differential equation integrate twice. In the latter case we get $y = \int(\int f(x)dx)dx + c_1x + c_2$.

64. Solving for y' using the quadratic formula we obtain the two differential equations

$$y' = \frac{1}{x} \left(2 + 2\sqrt{1 + 3x^6} \right) \quad \text{and} \quad y' = \frac{1}{x} \left(2 - 2\sqrt{1 + 3x^6} \right),$$

so the differential equation cannot be put in the form $dy/dx = f(x, y)$.

65. The differential equation $yy' - xy = 0$ has normal form $dy/dx = x$. These are not equivalent because $y = 0$ is a solution of the first differential equation but not a solution of the second.

66. Differentiating we get $y' = c_1 + 3c_2x^2$ and $y'' = 6c_2x$. Then $c_2 = y''/6x$ and $c_1 = y' - xy''/2$, so

$$y = \left(y' - \frac{xy''}{2}\right)x + \left(\frac{y''}{6x}\right)x^3 = xy' - \frac{1}{3}x^2y''$$

and the differential equation is $x^2y'' - 3xy' + 3y = 0$.

67. (a) Since e^{-x^2} is positive for all values of x , $dy/dx > 0$ for all x , and a solution, $y(x)$, of the differential equation must be increasing on any interval.

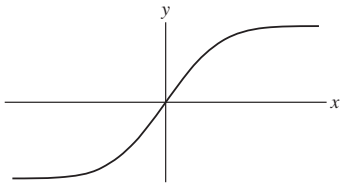
(b) $\lim_{x \rightarrow -\infty} \frac{dy}{dx} = \lim_{x \rightarrow -\infty} e^{-x^2} = 0$ and $\lim_{x \rightarrow \infty} \frac{dy}{dx} = \lim_{x \rightarrow \infty} e^{-x^2} = 0$. Since dy/dx approaches 0 as x approaches $-\infty$ and ∞ , the solution curve has horizontal asymptotes to the left and to the right.

(c) To test concavity we consider the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (e^{-x^2}) = -2xe^{-x^2}.$$

Since the second derivative is positive for $x < 0$ and negative for $x > 0$, the solution curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.

(d)



68. (a) The derivative of a constant solution $y = c$ is 0, so solving $5 - c = 0$ we see that $c = 5$ and so $y = 5$ is a constant solution.

(b) A solution is increasing where $dy/dx = 5 - y > 0$ or $y < 5$. A solution is decreasing where $dy/dx = 5 - y < 0$ or $y > 5$.

69. (a) The derivative of a constant solution is 0, so solving $y(a - by) = 0$ we see that $y = 0$ and $y = a/b$ are constant solutions.

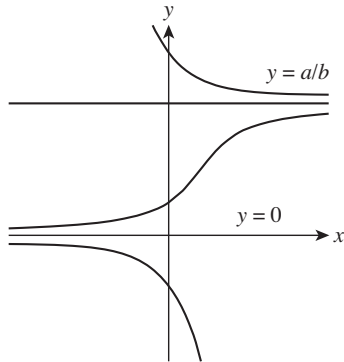
(b) A solution is increasing where $dy/dx = y(a - by) = by(a/b - y) > 0$ or $0 < y < a/b$. A solution is decreasing where $dy/dx = by(a/b - y) < 0$ or $y < 0$ or $y > a/b$.

(c) Using implicit differentiation we compute

$$\frac{d^2y}{dx^2} = y(-by') + y'(a - by) = y'(a - 2by).$$

Solving $d^2y/dx^2 = 0$ we obtain $y = a/2b$. Since $d^2y/dx^2 > 0$ for $0 < y < a/2b$ and $d^2y/dx^2 < 0$ for $a/2b < y < a/b$, the graph of $y = \phi(x)$ has a point of inflection at $y = a/2b$.

(d)

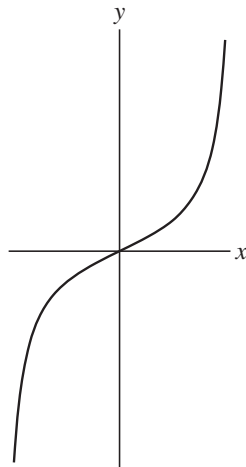


70. (a) If $y = c$ is a constant solution then $y' = 0$, but $c^2 + 4$ is never 0 for any real value of c .

(b) Since $y' = y^2 + 4 > 0$ for all x where a solution $y = \phi(x)$ is defined, any solution must be increasing on any interval on which it is defined. Thus it cannot have any relative extrema.

(c) Using implicit differentiation we compute $d^2y/dx^2 = 2yy' = 2y(y^2 + 4)$. Setting $d^2y/dx^2 = 0$ we see that $y = 0$ corresponds to the only possible point of inflection. Since $d^2y/dx^2 < 0$ for $y < 0$ and $d^2y/dx^2 > 0$ for $y > 0$, there is a point of inflection where $y = 0$.

(d)



71. In *Mathematica* use

```
Clear[y]
y[x_]:= x Exp[5x] Cos[2x]
y[x]
y''''[x] - 20y'''[x] + 158y''[x] - 580y'[x] + 841y[x]//Simplify
```

The output will show $y(x) = e^{5x}x \cos 2x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

72. In *Mathematica* use

```
Clear[y]
y[x_]:= 20Cos[5Log[x]]/x - 3Sin[5Log[x]]/x
y[x]
x^3 y'''[x] + 2x^2 y''[x] + 20x y'[x] - 78y[x]//Simplify
```

The output will show $y(x) = 20 \cos(5 \ln x)/x - 3 \sin(5 \ln x)/x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

1.2 Initial-Value Problems

1. Solving $-1/3 = 1/(1 + c_1)$ we get $c_1 = -4$. The solution is $y = 1/(1 - 4e^{-x})$.
2. Solving $2 = 1/(1 + c_1 e)$ we get $c_1 = -(1/2)e^{-1}$. The solution is $y = 2/(2 - e^{-(x+1)})$.
3. Letting $x = 2$ and solving $1/3 = 1/(4 + c)$ we get $c = -1$. The solution is $y = 1/(x^2 - 1)$. This solution is defined on the interval $(1, \infty)$.
4. Letting $x = -2$ and solving $1/2 = 1/(4 + c)$ we get $c = -2$. The solution is $y = 1/(x^2 - 2)$. This solution is defined on the interval $(-\infty, -\sqrt{2})$.
5. Letting $x = 0$ and solving $1 = 1/c$ we get $c = 1$. The solution is $y = 1/(x^2 + 1)$. This solution is defined on the interval $(-\infty, \infty)$.
6. Letting $x = 1/2$ and solving $-4 = 1/(1/4 + c)$ we get $c = -1/2$. The solution is $y = 1/(x^2 - 1/2) = 2/(2x^2 - 1)$. This solution is defined on the interval $(-1/\sqrt{2}, 1/\sqrt{2})$.

In Problems 7–10, we use $x = c_1 \cos t + c_2 \sin t$ and $x' = -c_1 \sin t + c_2 \cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2 .

7. From the initial conditions we obtain the system

$$c_1 = -1c_2 = 8$$

The solution of the initial-value problem is $x = -\cos t + 8\sin t$.

8. From the initial conditions we obtain the system

$$c_2 = 0 - c_1 = 1$$

The solution of the initial-value problem is $x = -\cos t$.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2} = 0$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

10. From the initial conditions we obtain

$$\begin{aligned}\frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 &= \sqrt{2} \\ -\frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 &= 2\sqrt{2}.\end{aligned}$$

Solving, we find $c_1 = -1$ and $c_2 = 3$. The solution of the initial-value problem is $x = -\cos t + 3\sin t$.

In Problems 11–14, we use $y = c_1e^x + c_2e^{-x}$ and $y' = c_1e^x - c_2e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2 .

11. From the initial conditions we obtain

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is $y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$.

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$

$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}e^x - \frac{1}{2}e^2e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}.$$

13. From the initial conditions we obtain

$$e^{-1}c_1 + ec_2 = 5$$

$$e^{-1}c_1 - ec_2 = -5.$$

Solving, we find $c_1 = 0$ and $c_2 = 5e^{-1}$. The solution of the initial-value problem is $y = 5e^{-1}e^{-x} = 5e^{-1-x}$.

14. From the initial conditions we obtain

$$c_1 + c_2 = 0$$

$$c_1 - c_2 = 0.$$

Solving, we find $c_1 = c_2 = 0$. The solution of the initial-value problem is $y = 0$.

15. Two solutions are $y = 0$ and $y = x^3$.

16. Two solutions are $y = 0$ and $y = x^2$. (Also, any constant multiple of x^2 is a solution.)

17. For $f(x, y) = y^{2/3}$ we have $\frac{\partial f}{\partial y} = \frac{2}{3}y^{-1/3}$. Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

18. For $f(x, y) = \sqrt{xy}$ we have $\partial f / \partial y = \frac{1}{2}\sqrt{x/y}$. Thus the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

19. For $f(x, y) = \frac{y}{x}$ we have $\frac{\partial f}{\partial y} = \frac{1}{x}$. Thus, the differential equation will have a unique solution in any region where $x \neq 0$.

20. For $f(x, y) = x + y$ we have $\frac{\partial f}{\partial y} = 1$. Thus, the differential equation will have a unique solution in the entire plane.

21. For $f(x, y) = x^2/(4 - y^2)$ we have $\partial f / \partial y = 2x^2y/(4 - y^2)^2$. Thus the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

22. For $f(x, y) = \frac{x^2}{1 + y^3}$ we have $\frac{\partial f}{\partial y} = \frac{-3x^2y^2}{(1 + y^3)^2}$. Thus, the differential equation will have a unique solution in any region where $y \neq -1$.

23. For $f(x, y) = \frac{y^2}{x^2 + y^2}$ we have $\frac{\partial f}{\partial y} = \frac{2x^2y}{(x^2 + y^2)^2}$. Thus, the differential equation will have a unique solution in any region not containing $(0, 0)$.

24. For $f(x, y) = (y + x)/(y - x)$ we have $\partial f/\partial y = -2x/(y - x)^2$. Thus the differential equation will have a unique solution in any region where $y < x$ or where $y > x$.

In Problems 25–28, we identify $f(x, y) = \sqrt{y^2 - 9}$ and $\partial f/\partial y = y/\sqrt{y^2 - 9}$. We see that f and $\partial f/\partial y$ are both continuous in the regions of the plane determined by $y < -3$ and $y > 3$ with no restrictions on x .

25. Since $4 > 3$, $(1, 4)$ is in the region defined by $y > 3$ and the differential equation has a unique solution through $(1, 4)$.
26. Since $(5, 3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(5, 3)$.
27. Since $(2, -3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(2, -3)$.
28. Since $(-1, 1)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(-1, 1)$.
29. (a) A one-parameter family of solutions is $y = cx$. Since $y' = c$, $xy' = xc = y$ and $y(0) = c \cdot 0 = 0$.

(b) Writing the equation in the form $y' = y/x$, we see that R cannot contain any point on the y -axis. Thus, any rectangular region disjoint from the y -axis and containing (x_0, y_0) will determine an interval around x_0 and a unique solution through (x_0, y_0) . Since $x_0 = 0$ in part (a), we are not guaranteed a unique solution through $(0, 0)$.

(c) The piecewise-defined function which satisfies $y(0) = 0$ is not a solution since it is not differentiable at $x = 0$.

30. (a) Since $\frac{d}{dx} \tan(x + c) = \sec^2(x + c) = 1 + \tan^2(x + c)$, we see that $y = \tan(x + c)$ satisfies the differential equation.

(b) Solving $y(0) = \tan c = 0$ we obtain $c = 0$ and $y = \tan x$. Since $\tan x$ is discontinuous at $x = \pm\pi/2$, the solution is not defined on $(-2, 2)$ because it contains $\pm\pi/2$.

(c) The largest interval on which the solution can exist is $(-\pi/2, \pi/2)$.

31. (a) Since $\frac{d}{dx} \left(-\frac{1}{x + c} \right) = \frac{1}{(x + c)^2} = y^2$, we see that $y = -\frac{1}{x + c}$ is a solution of the differential equation.

(b) Solving $y(0) = -1/c = 1$ we obtain $c = -1$ and $y = 1/(1 - x)$. Solving $y(0) = -1/c = -1$ we obtain $c = 1$ and $y = -1/(1 + x)$. Being sure to include $x = 0$, we see that the interval of existence of $y = 1/(1 - x)$ is $(-\infty, 1)$, while the interval of existence of $y = -1/(1 + x)$ is $(-1, \infty)$.

32. (a) Solving $y(0) = -1/c = y_0$ we obtain $c = -1/y_0$ and

$$y = -\frac{1}{-1/y_0 + x} = y_0(1 - y_0x), \quad y_0 \neq 0$$

Since we must have $-1/y_0 + x \neq 0$, the largest interval of existence (which must contain 0) is either $(-\infty, 1/y_0)$ when $y_0 > 0$ or $(1/y_0, \infty)$ when $y_0 < 0$.

- (b) By inspection we see that $y = 0$ is a solution on $(-\infty, \infty)$.

33. (a) Differentiating $3x^2 - y^2 = c$ we get $6x - 2yy' = 0$ or $yy' = 3x$.

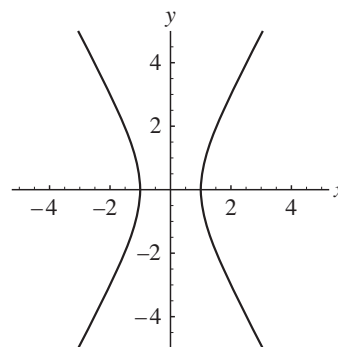
- (b) Solving $3x^2 - y^2 = 3$ for y we get

$$y = \phi_1(x) = \sqrt{3(x^2 - 1)}, \quad 1 < x < \infty,$$

$$y = \phi_2(x) = -\sqrt{3(x^2 - 1)}, \quad 1 < x < \infty,$$

$$y = \phi_3(x) = \sqrt{3(x^2 - 1)}, \quad -\infty < x < -1,$$

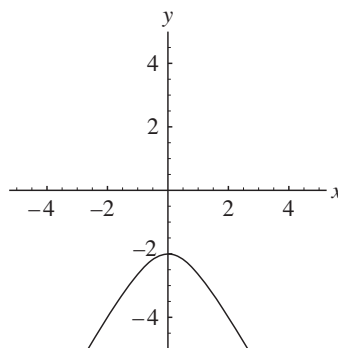
$$y = \phi_4(x) = -\sqrt{3(x^2 - 1)}, \quad -\infty < x < -1.$$



- (c) Only $y = \phi_3(x)$ satisfies $y(-2) = 3$.

34. (a) Setting $x = 2$ and $y = -4$ in $3x^2 - y^2 = c$ we get $12 - 16 = -4 = c$, so the explicit solution is

$$y = -\sqrt{3x^2 + 4}, \quad -\infty < x < \infty.$$



- (b) Setting $c = 0$ we have $y = \sqrt{3}x$ and $y = -\sqrt{3}x$, both defined on $(-\infty, \infty)$.

In Problems 35–38, we consider the points on the graphs with x -coordinates $x_0 = -1$, $x_0 = 0$, and $x_0 = 1$. The slopes of the tangent lines at these points are compared with the slopes given by $y'(x_0)$ in (a) through (f).

35. The graph satisfies the conditions in (b) and (f).

36. The graph satisfies the conditions in (e).

37. The graph satisfies the conditions in (c) and (d).

38. The graph satisfies the conditions in (a).

39. Using the function $y = c_1 \cos 3x + c_2 \sin 3x$ and the first boundary condition we get

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = 0$$

Therefore $c_1 = 0$. Similarly, for the second boundary condition we get

$$y(\pi/6) = c_2 \sin 3(\pi/6) = -1$$

Therefore $c_2 = -1$. The solution to the boundary value problem is $y(x) = -\sin 3x$.

40. Using the function $y = c_1 \cos 3x + c_2 \sin 3x$ and the first boundary condition we get

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = 0$$

Therefore $c_1 = 0$. Thus $y(x) = c_2 \sin 3x$. Similarly for the second boundary condition we get

$$y(\pi) = c_2 \sin 3(\pi) = 0$$

But the last line is satisfied for any choice of c_2 since $\sin 3\pi = 0$ therefore there are infinitely many solutions in this case.

41. The derivative of the function $y = c_1 \cos 3x + c_2 \sin 3x$ is $y' = -3c_1 \sin 3x + 3c_2 \cos 3x$ and using the first boundary condition we get

$$y'(0) = 0 + 3c_2 = 0$$

Therefore $c_2 = 0$. So far then we have $y' = -3c_1 \sin 3x$. Similarly, using the second boundary condition we get

$$y'(\pi/4) = -3c_1 \left(\sqrt{2}/2 \right) = 0$$

Therefore $c_1 = 0$. The solution is thus the trivial solution $y = 0$.

42. The derivative of the function $y = c_1 \cos 3x + c_2 \sin 3x$ is $y' = -3c_1 \sin 3x + 3c_2 \cos 3x$ and using the two boundary conditions we get

$$y(0) = c_1 + 0 = 1$$

Therefore $c_1 = 1$. In addition

$$y'(\pi) = 0 - 3c_2 = 5$$

Therefore $c_2 = -5/3$. The solution to this boundary value problem is $y(x) = \cos 3x - \frac{5}{3} \sin 3x$.

43. Using the function $y = c_1 \cos 3x + c_2 \sin 3x$ and the first boundary condition we get

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = 0$$

Therefore $c_1 = 0$. Thus $y(x) = c_2 \sin 3x$. Similarly for the second boundary condition we get

$$c_2 \sin 3(\pi) = 4c_2 \cdot 0 = 40 = 4$$

The last line is obviously a contradiction and so therefore the boundary value problem has no solution in this case.

44. The derivative of the function $y = c_1 \cos 3x + c_2 \sin 3x$ is $y' = -3c_1 \sin 3x + 3c_2 \cos 3x$ and using the first boundary condition we get

$$y'(\pi/3) = 0 - 3c_2 = 1$$

Therefore $c_2 = -1/3$. Similarly, using the second boundary condition we get

$$y'(\pi) = 0 - 3c_2 = 0$$

Therefore $c_2 = 0$, which is a contradiction so the problem has no solution.

45. Integrating $y' = 8e^{2x} + 6x$ we obtain

$$y = \int (8e^{2x} + 6x) dx = 4e^{2x} + 3x^2 + c.$$

Setting $x = 0$ and $y = 9$ we have $9 = 4 + c$ so $c = 5$ and $y = 4e^{2x} + 3x^2 + 5$.

46. Integrating $y'' = 12x - 2$ we obtain

$$y' = \int (12x - 2) dx = 6x^2 - 2x + c_1.$$

Then, integrating y' we obtain

$$y = \int (6x^2 - 2x + c_1) dx = 2x^3 - x^2 + c_1x + c_2.$$

At $x = 1$ the y -coordinate of the point of tangency is $y = -1 + 5 = 4$. This gives the initial condition $y(1) = 4$. The slope of the tangent line at $x = 1$ is $y'(1) = -1$. From the initial conditions we obtain

$$2 - 1 + c_1 + c_2 = 4 \quad \text{or} \quad c_1 + c_2 = 3$$

and

$$6 - 2 + c_1 = -1 \quad \text{or} \quad c_1 = -5.$$

Thus, $c_1 = -5$ and $c_2 = 8$, so $y = 2x^3 - x^2 - 5x + 8$.

47. When $x = 0$ and $y = \frac{1}{2}$, $y' = -1$, so the only plausible solution curve is the one with negative slope at $(0, \frac{1}{2})$, or the blue curve.
48. If $y(x)$ is a solution of the initial-value problem $y' = x^2 + y^2$. $y(0) = 1$, then $y'(0) = 0^2 + 1^2 = 1$. Differentiating the differential equation, we have $y'' = 2x + 2yy'$. So $y''(0) = 2 \cdot 0 + 2 \cdot 1 \cdot 1 = 2$.
49. If the solution is tangent to the x -axis at $(x_0, 0)$, then $y' = 0$ when $x = x_0$ and $y = 0$. Substituting these values into $y' + 2y = 3x - 6$ we get $0 + 0 = 3x_0 - 6$ or $x_0 = 2$.
50. When $y = \frac{1}{16}x^4$, $y' = \frac{1}{4}x^3 = x(\frac{1}{4}x^2) = xy^{1/2}$, and $y(2) = \frac{1}{16}(16) = 1$. When

$$y = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

we have

$$y' = \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^3, & x \geq 0 \end{cases} = x \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^2, & x \geq 0 \end{cases} = xy^{1/2},$$

and $y(2) = \frac{1}{16}(16) = 1$. The two different solutions are the same on the interval $(0, \infty)$, which is all that is required by Theorem 1.2.1.

51. We note that the initial condition $y(0) = 0$,

$$0 = \int_0^y \frac{1}{\sqrt{t^3 + 1}} dt$$

is satisfied only when $y = 0$. For any $y > 0$, necessarily

$$\int_0^y \frac{1}{\sqrt{t^3 + 1}} dt > 0$$

because the integrand is positive on the interval of integration. Then from (12) of Section 1.1 and the Chain Rule we have:

$$\begin{aligned} \frac{d}{dx}x &= \frac{d}{dx} \int_0^y \frac{1}{\sqrt{t^3 + 1}} dt & \frac{dy}{dx} &= \sqrt{y^3 + 1} \\ 1 &= \frac{1}{\sqrt{y^3 + 1}} \frac{dy}{dx} & \text{and} & \\ y'(0) &= \frac{dy}{dx} \Big|_{x=0} = \sqrt{(y(0))^3 + 1} = \sqrt{0 + 1} = 1. \end{aligned}$$

Computing the second derivative, we see that:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \sqrt{y^3 + 1} = \frac{3y^2}{2\sqrt{y^3 + 1}} \frac{dy}{dx} = \frac{3y^2}{2\sqrt{y^3 + 1}} \cdot \sqrt{y^3 + 1} = \frac{3}{2}y^2 \\ \frac{d^2y}{dx^2} &= \frac{3}{2}y^2. \end{aligned}$$

This is equivalent to $2\frac{d^2y}{dx^2} - 3y^2 = 0$.

1.3

Differential Equations as Mathematical Models

1. $\frac{dP}{dt} = kP + r; \quad \frac{dP}{dt} = kP - r$
2. Let b be the rate of births and d the rate of deaths. Then $b = k_1P$ and $d = k_2P$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P$.
3. Let b be the rate of births and d the rate of deaths. Then $b = k_1P$ and $d = k_2P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P^2$.
4. $\frac{dP}{dt} = k_1P - k_2P^2 - h, \quad h > 0$
5. From the graph in the text we estimate $T_0 = 180^\circ$ and $T_m = 75^\circ$. We observe that when $T = 85$, $dT/dt \approx -1$. From the differential equation we then have

$$k = \frac{dT/dt}{T - T_m} = \frac{-1}{85 - 75} = -0.1.$$

6. By inspecting the graph in the text we take T_m to be $T_m(t) = 80 - 30 \cos(\pi t/12)$. Then the temperature of the body at time t is determined by the differential equation

$$\frac{dT}{dt} = k \left[T - \left(80 - 30 \cos \left(\frac{\pi}{12} t \right) \right) \right], \quad t > 0.$$

7. The number of students with the flu is x and the number not infected is $1000 - x$, so $dx/dt = kx(1000 - x)$.
8. By analogy, with the differential equation modeling the spread of a disease, we assume that the rate at which the technological innovation is adopted is proportional to the number of people who have adopted the innovation and also to the number of people, $y(t)$, who have not yet adopted it. If one person who has adopted the innovation is introduced into the population, then $x + y = n + 1$ and

$$\frac{dx}{dt} = kx(n + 1 - x), \quad x(0) = 1.$$

9. The rate at which salt is leaving the tank is

$$R_{out} (3 \text{ gal/min}) \cdot \left(\frac{A}{300} \text{ lb/gal} \right) = \frac{A}{100} \text{ lb/min}.$$

Thus $dA/dt = A/100$. The initial amount is $A(0) = 50$.

10. The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min.}$$

Since the solution is pumped out at a slower rate, it is accumulating at the rate of $(3 - 2)$ gal/min = 1 gal/min. After t minutes there are $300 + t$ gallons of brine in the tank. The rate at which salt is leaving is

$$R_{out} = (2 \text{ gal/min}) \cdot \left(\frac{A}{300 + t} \text{ lb/gal} \right) = \frac{2A}{300 + t} \text{ lb/min.}$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{2A}{300 + t}.$$

11. The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min.}$$

Since the tank loses liquid at the net rate of

$$3 \text{ gal/min} - 3.5 \text{ gal/min} = -0.5 \text{ gal/min,}$$

after t minutes the number of gallons of brine in the tank is $300 - \frac{1}{2}t$ gallons. Thus the rate at which salt is leaving is

$$R_{out} = \left(\frac{A}{300 - t/2} \text{ lb/gal} \right) \cdot (3.5 \text{ gal/min}) = \frac{3.5A}{300 - t/2} \text{ lb/min} = \frac{7A}{600 - t} \text{ lb/min.}$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{7A}{600 - t} \quad \text{or} \quad \frac{dA}{dt} + \frac{7}{600 - t} A = 6.$$

12. The rate at which salt is entering the tank is

$$R_{in} = (c_{in} \text{ lb/gal}) \cdot (r_{in} \text{ gal/min}) = c_{in}r_{in} \text{ lb/min.}$$

Now let $A(t)$ denote the number of pounds of salt and $N(t)$ the number of gallons of brine in the tank at time t . The concentration of salt in the tank as well as in the outflow is $c(t) = x(t)/N(t)$. But the number of gallons of brine in the tank remains steady, is increased, or is decreased depending on whether $r_{in} = r_{out}$, $r_{in} > r_{out}$, or $r_{in} < r_{out}$. In any case, the number of gallons of brine in the tank at time t is $N(t) = N_0 + (r_{in} - r_{out})t$. The output rate of salt is then

$$R_{out} = \left(\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/gal} \right) \cdot (r_{out} \text{ gal/min}) = r_{out} \frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/min.}$$

The differential equation for the amount of salt, $dA/dt = R_{in} - R_{out}$, is

$$\frac{dA}{dt} = c_{in}r_{in} - r_{out} \frac{A}{N_0 + (r_{in} - r_{out})t} \quad \text{or} \quad \frac{dA}{dt} + \frac{r_{out}}{N_0 + (r_{in} - r_{out})t} A = c_{in}r_{in}.$$

13. The volume of water in the tank at time t is $V = A_w h$. The differential equation is then

$$\frac{dh}{dt} = \frac{1}{A_w} \frac{dV}{dt} = \frac{1}{A_w} \left(-c A_h \sqrt{2gh} \right) = -\frac{c A_h}{A_w} \sqrt{2gh}.$$

Using $A_h = \pi \left(\frac{2}{12} \right)^2 = \frac{\pi}{36}$, $A_w = 10^2 = 100$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/36}{100} \sqrt{64h} = -\frac{c\pi}{450} \sqrt{h}.$$

14. The volume of water in the tank at time t is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the tank at height h . From the figure in the text we see that $r/h = 8/20$ so that $r = \frac{2}{5}h$ and $V = \frac{1}{3}\pi \left(\frac{2}{5}h \right)^2 h = \frac{4}{75}\pi h^3$. Differentiating with respect to t we have $dV/dt = \frac{4}{25}\pi h^2 dh/dt$ or

$$\frac{dh}{dt} = \frac{25}{4\pi h^2} \frac{dV}{dt}.$$

From Problem 13 we have $dV/dt = -c A_h \sqrt{2gh}$ where $c = 0.6$, $A_h = \pi \left(\frac{2}{12} \right)^2$, and $g = 32$. Thus $dV/dt = -2\pi\sqrt{h}/15$ and

$$\frac{dh}{dt} = \frac{25}{4\pi h^2} \left(-\frac{2\pi\sqrt{h}}{15} \right) = -\frac{5}{6h^{3/2}}.$$

15. Since $i = dq/dt$ and $L d^2q/dt^2 + R dq/dt = E(t)$, we obtain $L di/dt + Ri = E(t)$.

16. By Kirchhoff's second law we obtain $R \frac{dq}{dt} + \frac{1}{C} q = E(t)$.

17. From Newton's second law we obtain $m \frac{dv}{dt} = -kv^2 + mg$.

18. Since the barrel in Figure 1.3.17(b) in the text is submerged an additional y feet below its equilibrium position, the number of cubic feet in the additional submerged portion is the volume of the circular cylinder: $\pi \times (\text{radius})^2 \times \text{height}$ or $\pi(s/2)^2 y$. Then we have from Archimedes' principle

$$\begin{aligned} \text{Upward force of water on barrel} &= \text{Weight of water displaced} \\ &= (62.4) \times (\text{Volume of water displaced}) \\ &= (62.4)\pi(s/2)^2 y = 15.6\pi s^2 y. \end{aligned}$$

It then follows from Newton's second law that

$$\frac{w}{g} \frac{d^2 y}{dt^2} = -15.6\pi s^2 y \quad \text{or} \quad \frac{d^2 y}{dt^2} + \frac{15.6\pi s^2 g}{w} y = 0,$$

where $g = 32$ and w is the weight of the barrel in pounds.

19. The net force acting on the mass is

$$F = ma = m \frac{d^2x}{dt^2} = -k(s + x) + mg = -kx + mg - ks.$$

Since the condition of equilibrium is $mg = ks$, the differential equation is

$$m \frac{d^2x}{dt^2} = -kx.$$

20. From Problem 19, without a damping force, the differential equation is $m \frac{d^2x}{dt^2} = -kx$.
With a damping force proportional to velocity, the differential equation becomes

$$m \frac{d^2x}{dt^2} = -kx - \beta \frac{dx}{dt} \quad \text{or} \quad m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = 0.$$

21. As the rocket climbs (in the positive direction), it spends its amount of fuel and therefore the mass of the fuel changes with time. The air resistance acts in the opposite direction of the motion and the upward thrust R works in the same direction. Using Newton's second law we get

$$\frac{d}{dt}(mv) = -mg - kv + R$$

Now because the mass is variable, we must use the product rule to expand the left side of the equation. Doing so gives us the following:

$$\begin{aligned} \frac{d}{dt}(mv) &= -mg - kv + R \\ v \times \frac{dm}{dt} + m \times \frac{dv}{dt} &= -mg - kv + R \end{aligned}$$

The last line is the differential equation we wanted to find.

22. (a) Since the mass of the rocket is $m(t) = m_p + m_v + m_f(t)$, take the time rate-of-change and get, by straight-forward calculation,

$$\frac{d}{dt}m(t) = \frac{d}{dt}(m_p + m_v + m_f(t)) = 0 + 0 + m'_f(t) = \frac{d}{dt}m_f(t)$$

Therefore the rate of change of the mass of the rocket is the same as the rate of change of the mass of the fuel which is what we wanted to show.

- (b) The fuel is decreasing at the constant rate of λ and so from part (a) we have

$$\begin{aligned} \frac{d}{dt}m(t) &= \frac{d}{dt}m_f(t) = -\lambda \\ m(t) &= -\lambda t + c \end{aligned}$$

Using the given condition to solve for c , $m(0) = 0 + c = m_0$ and so $m(t) = -\lambda t + m_0$. The differential equation in Problem 21 now becomes

$$\begin{aligned} v \frac{dm}{dt} + m \frac{dv}{dt} + kv &= -mg + R \\ -\lambda v + (-\lambda t + m_0) \frac{dv}{dt} + kv &= -mg + R \\ (-\lambda t + m_0) \frac{dv}{dt} + (k - \lambda)v &= -mg + R \\ \frac{dv}{dt} + \frac{k - \lambda}{-\lambda t + m_0} v &= \frac{-mg}{-\lambda t + m_0} + \frac{R}{-\lambda t + m_0} \\ \frac{dv}{dt} + \frac{k - \lambda}{-\lambda t + m_0} v &= -g + \frac{R}{-\lambda t + m_0} \end{aligned}$$

(c) From part (b) we have that $\frac{d}{dt}m_f(t) = -\lambda$ and so by integrating this result we get $m_f(t) = -\lambda t + c$. Now at time $t = 0$, $m_f(0) = 0 + c = c$ therefore $m_f(t) = -\lambda t + m_f(0)$. At some later time t_b we then have $m_f(t_b) = -\lambda t_b + m_f(0) = 0$ and solving this equation for that time we get $t_b = m_f(0)/\lambda$ which is what we wanted to show.

23. From $g = k/R^2$ we find $k = gR^2$. Using $a = d^2r/dt^2$ and the fact that the positive direction is upward we get

$$\frac{d^2r}{dt^2} = -a = -\frac{k}{r^2} = -\frac{gR^2}{r^2} \quad \text{or} \quad \frac{d^2r}{dt^2} + \frac{gR^2}{r^2} = 0.$$

24. The gravitational force on m is $F = -kM_r m/r^2$. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$, we have $M_r = r^3M/R^3$ and

$$F = -k \frac{M_r m}{r^2} = -k \frac{r^3 M m / R^3}{r^2} = -k \frac{mM}{R^3} r.$$

Now from $F = ma = d^2r/dt^2$ we have

$$m \frac{d^2r}{dt^2} = -k \frac{mM}{R^3} r \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{kM}{R^3} r.$$

25. The differential equation is $\frac{dA}{dt} = k(M - A)$.

26. The differential equation is $\frac{dA}{dt} = k_1(M - A) - k_2A$.

27. The differential equation is $x'(t) = r - kx(t)$ where $k > 0$.

28. Notice from the diagram that the segment from the waterskier to the boat is tangent to the curve at the point $P(x, y)$. For the sake of simplicity, let's label the coordinates of the boat on the x -axis as $(a, 0)$. The equation of this tangent line is $y = y'(x)(x - a)$ from which we get

$$\begin{aligned} y &= y'(x)(x - a) \\ \frac{y}{y'} &= x - a \\ -\frac{y}{y'} &= a - x \end{aligned}$$

By the Pythagorean Theorem we have $y^2 + (a - x)^2 = s^2$ from which we make a substitution and then solve for the derivative to get

$$\begin{aligned} y^2 + (a - x)^2 &= s^2 \\ y^2 + (y/y')^2 &= s^2 \\ (y/y')^2 &= s^2 - y^2 \\ \frac{y}{y'} &= -\sqrt{s^2 - y^2} \\ y' &= -\frac{y}{\sqrt{s^2 - y^2}} \end{aligned}$$

Notice that the sign of the derivative is negative because as the boat proceeds along the positive x -axis, the y -coordinate decreases.

29. We see from the figure that $2\theta + \alpha = \pi$. Thus

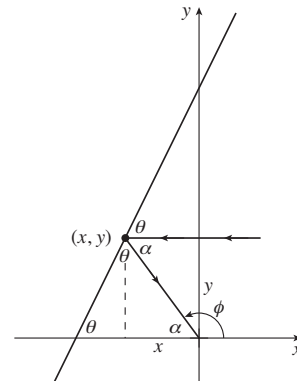
$$\frac{y}{-x} = \tan \alpha = \tan(\pi - 2\theta) = -\tan 2\theta = -\frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

Since the slope of the tangent line is $y' = \tan \theta$ we have $y/x = 2y'[1 - (y')^2]$ or $y - y(y')^2 = 2xy'$, which is the quadratic equation $y(y')^2 + 2xy' - y = 0$ in y' . Using the quadratic formula, we get

$$y' = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}.$$

Since $dy/dx > 0$, the differential equation is

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y} \quad \text{or} \quad y \frac{dy}{dx} - \sqrt{x^2 + y^2} + x = 0.$$



30. The differential equation is $dP/dt = kP$, so from Problem 49 in Exercises 1.1, a one-parameter family of solutions is $P = ce^{kt}$.
31. The differential equation in (3) is $dT/dt = k(T - T_m)$. When the body is cooling, $T > T_m$, so $T - T_m > 0$. Since T is decreasing, $dT/dt < 0$ and $k < 0$. When the body is warming, $T < T_m$, so $T - T_m < 0$. Since T is increasing, $dT/dt > 0$ and $k < 0$.
32. The differential equation in (8) is $dA/dt = 6 - A/100$. If $A(t)$ attains a maximum, then $dA/dt = 0$ at this time and $A = 600$. If $A(t)$ continues to increase without reaching a maximum, then $A'(t) > 0$ for $t > 0$ and A cannot exceed 600. In this case, if $A'(t)$ approaches 0 as t increases to infinity, we see that $A(t)$ approaches 600 as t increases to infinity.
33. This differential equation could describe a population that undergoes periodic fluctuations.
34. (a) As shown in Figure 1.3.23(a) in the text, the resultant of the reaction force of magnitude F and the weight of magnitude mg of the particle is the centripetal force of magnitude $m\omega^2x$. The centripetal force points to the center of the circle of radius x on which the particle rotates about the y -axis. Comparing parts of similar triangles gives

$$F \cos \theta = mg \quad \text{and} \quad F \sin \theta = m\omega^2x.$$

(b) Using the equations in part (a) we find

$$\tan \theta = \frac{F \sin \theta}{F \cos \theta} = \frac{m\omega^2x}{mg} = \frac{\omega^2x}{g} \quad \text{or} \quad \frac{dy}{dx} = \frac{\omega^2x}{g}.$$

35. From Problem 23, $d^2r/dt^2 = -gR^2/r^2$. Since R is a constant, if $r = R + s$, then $d^2r/dt^2 = d^2s/dt^2$ and, using a Taylor series, we get

$$\frac{d^2s}{dt^2} = -g \frac{R^2}{(R+s)^2} = -gR^2(R+s)^{-2} \approx -gR^2[R^{-2} - 2sR^{-3} + \cdots] = -g + \frac{2gs}{R} + \cdots.$$

Thus, for R much larger than s , the differential equation is approximated by $d^2s/dt^2 = -g$.

36. (a) If ρ is the mass density of the raindrop, then $m = \rho V$ and

$$\frac{dm}{dt} = \rho \frac{dV}{dt} = \rho \frac{d}{dt} \left[\frac{4}{3} \pi r^3 \right] = \rho \left(4\pi r^2 \frac{dr}{dt} \right) = \rho S \frac{dr}{dt}.$$

If dr/dt is a constant, then $dm/dt = kS$ where $\rho dr/dt = k$ or $dr/dt = k/\rho$. Since the radius is decreasing, $k < 0$. Solving $dr/dt = k/\rho$ we get $r = (k/\rho)t + c_0$. Since $r(0) = r_0$, $c_0 = r_0$ and $r = kt/\rho + r_0$.

- (b) From Newton's second law, $\frac{d}{dt}[mv] = mg$, where v is the velocity of the raindrop. Then

$$m \frac{dv}{dt} + v \frac{dm}{dt} = mg \quad \text{or} \quad \rho \left(\frac{4}{3} \pi r^3 \right) \frac{dv}{dt} + v(k4\pi r^2) = \rho \left(\frac{4}{3} \pi r^3 \right) g.$$

Dividing by $4\rho\pi r^3/3$ we get

$$\frac{dv}{dt} + \frac{3k}{\rho r} v = g \quad \text{or} \quad \frac{dv}{dt} + \frac{3k/\rho}{kt/\rho + r_0} v = g, \quad k < 0.$$

- 37.** We assume that the plow clears snow at a constant rate of k cubic miles per hour. Let t be the time in hours after noon, $x(t)$ the depth in miles of the snow at time t , and $y(t)$ the distance the plow has moved in t hours. Then dy/dt is the velocity of the plow and the assumption gives

$$wx \frac{dy}{dt} = k,$$

where w is the width of the plow. Each side of this equation simply represents the volume of snow plowed in one hour. Now let t_0 be the number of hours before noon when it started snowing and let s be the constant rate in miles per hour at which x increases. Then for $t > -t_0$, $x = s(t + t_0)$. The differential equation then becomes

$$\frac{dy}{dt} = \frac{k}{ws} \frac{1}{t + t_0}.$$

Integrating, we obtain

$$y = \frac{k}{ws} [\ln(t + t_0) + c]$$

where c is a constant. Now when $t = 0$, $y = 0$ so $c = -\ln t_0$ and

$$y = \frac{k}{ws} \ln \left(1 + \frac{t}{t_0} \right).$$

Finally, from the fact that when $t = 1$, $y = 2$ and when $t = 2$, $y = 3$, we obtain

$$\left(1 + \frac{2}{t_0} \right)^2 = \left(1 + \frac{1}{t_0} \right)^3.$$

Expanding and simplifying gives $t_0^2 + t_0 - 1 = 0$. Since $t_0 > 0$, we find $t_0 \approx 0.618$ hours ≈ 37 minutes. Thus it started snowing at about 11:23 in the morning.

- 38.** (1) : $\frac{dP}{dt} = kP$ is linear (2) : $\frac{dA}{dt} = kA$ is linear
 (3) : $\frac{dT}{dt} = k(T - T_m)$ is linear (5) : $\frac{dx}{dt} = kx(n + 1 - x)$ is nonlinear
 (6) : $\frac{dX}{dt} = k(\alpha - X)(\beta - X)$ is nonlinear (8) : $\frac{dA}{dt} = 6 - \frac{A}{100}$ is linear
 (10) : $\frac{dh}{dt} = -\frac{A_h}{A_w} \sqrt{2gh}$ is nonlinear (11) : $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C}q = E(t)$ is linear
 (12) : $\frac{d^2s}{dt^2} = -g$ is linear (14) : $m \frac{dv}{dt} = mg - kv$ is linear
 (15) : $m \frac{d^2s}{dt^2} + k \frac{ds}{dt} = mg$ is linear (16) : $\frac{d^2x}{dt^2} - \frac{64}{L}x = 0$ is linear
 (17) : linearity or nonlinearity is determined by the manner in which W and T_1 involve x .

- 39.** At time t , when the population is 2 million cells, the differential equation $P'(t) = 0.15P(t)$ gives the rate of increase at time t . Thus, when $P(t) = 2$ (million cells), the rate of increase is $P'(t) = 0.15(2) = 0.3$ million cells per hour or 300,000 cells per hour.

40. Setting $A'(t) = -0.002$ and solving $A'(t) = -0.0004332A(t)$ for $A(t)$, we obtain

$$A(t) = \frac{A'(t)}{-0.0004332} = \frac{-0.002}{-0.0004332} \approx 4.6 \text{ grams.}$$

Chapter 1 in Review

1. $\frac{d}{dx} c_1 e^{kx} = k \overbrace{c_1 e^{kx}}^y; \quad \frac{dy}{dx} = ky$
2. $\frac{d}{dx} (5 + c_1 e^{-2x}) = -2c_1 e^{-2x} = -2(\overbrace{5 + c_1 e^{-2x}}^y - 5); \quad \frac{dy}{dx} = -2(y - 5) \quad \text{or} \quad \frac{dy}{dx} = -2y + 10$
3. $\frac{d}{dx} (c_1 \cos kx + c_2 \sin kx) = -kc_1 \sin kx + kc_2 \cos kx;$
 $\frac{d^2}{dx^2} (c_1 \cos kx + c_2 \sin kx) = -k^2 c_1 \cos kx - k^2 c_2 \sin kx = -k^2 (\overbrace{c_1 \cos kx + c_2 \sin kx}^y);$
 $\frac{d^2 y}{dx^2} = -k^2 y \quad \text{or} \quad \frac{d^2 y}{dx^2} + k^2 y = 0$
4. $\frac{d}{dx} (c_1 \cosh kx + c_2 \sinh kx) = kc_1 \sinh kx + kc_2 \cosh kx;$
 $\frac{d^2}{dx^2} (c_1 \cosh kx + c_2 \sinh kx) = k^2 c_1 \cosh kx + k^2 c_2 \sinh kx = k^2 (\overbrace{c_1 \cosh kx + c_2 \sinh kx}^y);$
 $\frac{d^2 y}{dx^2} = k^2 y \quad \text{or} \quad \frac{d^2 y}{dx^2} - k^2 y = 0$
5. $y = c_1 e^x + c_2 x e^x; \quad y' = c_1 e^x + c_2 x e^x + c_2 e^x; \quad y'' = c_1 e^x + c_2 x e^x + 2c_2 e^x;$
 $y'' + y = 2(c_1 e^x + c_2 x e^x) + 2c_2 e^x = 2(c_1 e^x + c_2 x e^x + c_2 e^x) = 2y'; \quad y'' - 2y' + y = 0$
6. $y' = -c_1 e^x \sin x + c_1 e^x \cos x + c_2 e^x \cos x + c_2 e^x \sin x;$
 $y'' = -c_1 e^x \cos x - c_1 e^x \sin x - c_1 e^x \sin x + c_1 e^x \cos x - c_2 e^x \sin x + c_2 e^x \cos x + c_2 e^x \cos x + c_2 e^x \sin x$
 $= -2c_1 e^x \sin x + 2c_2 e^x \cos x;$
 $y'' - 2y' = -2c_1 e^x \cos x - 2c_2 e^x \sin x = -2y; \quad y'' - 2y' + 2y = 0$
7. a, d 8. c 9. b 10. a, c 11. b 12. a, b, d
13. A few solutions are $y = 0$, $y = c$, and $y = e^x$.
14. Easy solutions to see are $y = 0$ and $y = 3$.
15. The slope of the tangent line at (x, y) is y' , so the differential equation is $y' = x^2 + y^2$.

16. The rate at which the slope changes is $dy'/dx = y''$, so the differential equation is $y'' = -y'$ or $y'' + y' = 0$.

17. (a) The domain is all real numbers.

(b) Since $y' = 2/3x^{1/3}$, the solution $y = x^{2/3}$ is undefined at $x = 0$. This function is a solution of the differential equation on $(-\infty, 0)$ and also on $(0, \infty)$.

18. (a) Differentiating $y^2 - 2y = x^2 - x + c$ we obtain $2yy' - 2y' = 2x - 1$ or $(2y - 2)y' = 2x - 1$.

(b) Setting $x = 0$ and $y = 1$ in the solution we have $1 - 2 = 0 - 0 + c$ or $c = -1$. Thus, a solution of the initial-value problem is $y^2 - 2y = x^2 - x - 1$.

(c) Solving the equation $y^2 - 2y - (x^2 - x - 1) = 0$ by the quadratic formula we get $y = (2 \pm \sqrt{4 + 4(x^2 - x - 1)})/2 = 1 \pm \sqrt{x^2 - x} = 1 \pm \sqrt{x(x - 1)}$. Since $x(x - 1) \geq 0$ for $x \leq 0$ or $x \geq 1$, we see that neither $y = 1 + \sqrt{x(x - 1)}$ nor $y = 1 - \sqrt{x(x - 1)}$ is differentiable at $x = 0$. Thus, both functions are solutions of the differential equation, but neither is a solution of the initial-value problem.

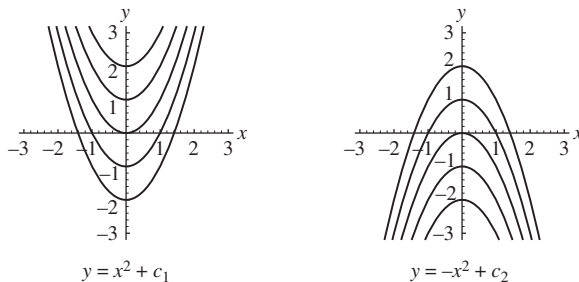
19. Setting $x = x_0$ and $y = 1$ in $y = -2/x + x$, we get

$$1 = -\frac{2}{x_0} + x_0 \quad \text{or} \quad x_0^2 - x_0 - 2 = (x_0 - 2)(x_0 + 1) = 0.$$

Thus, $x_0 = 2$ or $x_0 = -1$. Since $x = 0$ in $y = -2/x + x$, we see that $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(-1) = 1$, on the interval $(-\infty, 0)$ and $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(2) = 1$, on the interval $(0, \infty)$.

20. From the differential equation, $y'(1) = 1^2 + [y(1)]^2 = 1 + (-1)^2 = 2 > 0$, so $y(x)$ is increasing in some neighborhood of $x = 1$. From $y'' = 2x + 2yy'$ we have $y''(1) = 2(1) + 2(-1)(2) = -2 < 0$, so $y(x)$ is concave down in some neighborhood of $x = 1$.

21. (a)



(b) When $y = x^2 + c_1$, $y' = 2x$ and $(y')^2 = 4x^2$. When $y = -x^2 + c_2$, $y' = -2x$ and $(y')^2 = 4x^2$.

(c) Pasting together x^2 , $x \geq 0$, and $-x^2$, $x \leq 0$, we get $y = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0. \end{cases}$

22. The slope of the tangent line is $y'|_{(-1,4)} = 6\sqrt{4} + 5(-1)^3 = 7$.

23. Differentiating $y = x \sin x + x \cos x$ we get

$$y' = x \cos x + \sin x - x \sin x + \cos x$$

and

$$\begin{aligned} y'' &= -x \sin x + \cos x + \cos x - x \cos x - \sin x - \sin x \\ &= -x \sin x - x \cos x + 2 \cos x - 2 \sin x. \end{aligned}$$

Thus

$$y'' + y = -x \sin x - x \cos x + 2 \cos x - 2 \sin x + x \sin x + x \cos x = 2 \cos x - 2 \sin x.$$

An interval of definition for the solution is $(-\infty, \infty)$.

24. Differentiating $y = x \sin x + (\cos x) \ln(\cos x)$ we get

$$\begin{aligned} y' &= x \cos x + \sin x + \cos x \left(\frac{-\sin x}{\cos x} \right) - (\sin x) \ln(\cos x) \\ &= x \cos x + \sin x - \sin x - (\sin x) \ln(\cos x) \\ &= x \cos x - (\sin x) \ln(\cos x) \end{aligned}$$

and

$$\begin{aligned} y'' &= -x \sin x + \cos x - \sin x \left(\frac{-\sin x}{\cos x} \right) - (\cos x) \ln(\cos x) \\ &= -x \sin x + \cos x + \frac{\sin^2 x}{\cos x} - (\cos x) \ln(\cos x) \\ &= -x \sin x + \cos x + \frac{1 - \cos^2 x}{\cos x} - (\cos x) \ln(\cos x) \\ &= -x \sin x + \cos x + \sec x - \cos x - (\cos x) \ln(\cos x) \\ &= -x \sin x + \sec x - (\cos x) \ln(\cos x). \end{aligned}$$

Thus

$$y'' + y = -x \sin x + \sec x - (\cos x) \ln(\cos x) + x \sin x + (\cos x) \ln(\cos x) = \sec x.$$

To obtain an interval of definition we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos x > 0$. Thus, an interval of definition is $(-\pi/2, \pi/2)$.

25. Differentiating $y = \sin(\ln x)$ we obtain $y' = \cos(\ln x)/x$ and $y'' = -[\sin(\ln x) + \cos(\ln x)]/x^2$.
Then

$$x^2 y'' + xy' + y = x^2 \left(-\frac{\sin(\ln x) + \cos(\ln x)}{x^2} \right) + x \frac{\cos(\ln x)}{x} + \sin(\ln x) = 0.$$

An interval of definition for the solution is $(0, \infty)$.

26. Differentiating $y = \cos(\ln x) \ln(\cos(\ln x)) + (\ln x) \sin(\ln x)$ we obtain

$$\begin{aligned} y' &= \cos(\ln x) \frac{1}{\cos(\ln x)} \left(-\frac{\sin(\ln x)}{x} \right) + \ln(\cos(\ln x)) \left(-\frac{\sin(\ln x)}{x} \right) \\ &\quad + \ln x \frac{\cos(\ln x)}{x} + \frac{\sin(\ln x)}{x} \\ &= -\frac{\ln(\cos(\ln x)) \sin(\ln x)}{x} + \frac{(\ln x) \cos(\ln x)}{x} \end{aligned}$$

and

$$\begin{aligned} y'' &= -x \left[\ln(\cos(\ln x)) \frac{\cos(\ln x)}{x} + \sin(\ln x) \frac{1}{\cos(\ln x)} \left(-\frac{\sin(\ln x)}{x} \right) \right] \frac{1}{x^2} \\ &\quad + \ln(\cos(\ln x)) \sin(\ln x) \frac{1}{x^2} + x \left[(\ln x) \left(-\frac{\sin(\ln x)}{x} \right) + \frac{\cos(\ln x)}{x} \right] \frac{1}{x^2} \\ &\quad - (\ln x) \cos(\ln x) \frac{1}{x^2} \\ &= \frac{1}{x^2} \left[-\ln(\cos(\ln x)) \cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x)) \sin(\ln x) \right. \\ &\quad \left. - (\ln x) \sin(\ln x) + \cos(\ln x) - (\ln x) \cos(\ln x) \right]. \end{aligned}$$

Then

$$\begin{aligned} x^2 y'' + xy' + y &= -\ln(\cos(\ln x)) \cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x)) \sin(\ln x) \\ &\quad - (\ln x) \sin(\ln x) + \cos(\ln x) - (\ln x) \cos(\ln x) - \ln(\cos(\ln x)) \sin(\ln x) \\ &\quad + (\ln x) \cos(\ln x) + \cos(\ln x) \ln(\cos(\ln x)) + (\ln x) \sin(\ln x) \\ &= \frac{\sin^2(\ln x)}{\cos(\ln x)} + \cos(\ln x) = \frac{\sin^2(\ln x) + \cos^2(\ln x)}{\cos(\ln x)} = \frac{1}{\cos(\ln x)} = \sec(\ln x). \end{aligned}$$

To obtain an interval of definition, we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos(\ln x) > 0$. Since $\cos x > 0$ when $-\pi/2 < x < \pi/2$, we require $-\pi/2 < \ln x < \pi/2$. Since e^x is an increasing function, this is equivalent to $e^{-\pi/2} < x < e^{\pi/2}$. Thus, an interval of definition is $(e^{-\pi/2}, e^{\pi/2})$. (Much of this problem is more easily done using a computer algebra system such as *Mathematica* or *Maple*.)

In Problems 27–30 we use (12) of Section 1.1 and the Product Rule.

27.

$$y = e^{\cos x} \int_0^x t e^{-\cos t} dt$$

$$\frac{dy}{dx} = e^{\cos x} (x e^{-\cos x}) - \sin x e^{\cos x} \int_0^x t e^{-\cos t} dt$$

$$\begin{aligned} \frac{dy}{dx} + (\sin x) y &= e^{\cos x} x e^{-\cos x} - \sin x e^{\cos x} \int_0^x t e^{-\cos t} dt + \sin x \left(e^{\cos x} \int_0^x t e^{-\cos t} dt \right) \\ &= x - \sin x e^{\cos x} \int_0^x t e^{-\cos t} dt + \sin x e^{\cos x} \int_0^x t e^{-\cos t} dt = x \end{aligned}$$

28.

$$y = e^{x^2} \int_0^x e^{t-t^2} dt$$

$$\frac{dy}{dx} = e^{x^2} e^{x-x^2} + 2x e^{x^2} \int_0^x e^{t-t^2} dt$$

$$\frac{dy}{dx} - 2xy = e^{x^2} e^{x-x^2} + 2x e^{x^2} \int_0^x e^{t-t^2} dt - 2x \left(e^{x^2} \int_0^x e^{t-t^2} dt \right) = e^x$$

29.

$$y = x \int_1^x \frac{e^{-t}}{t} dt$$

$$y' = x \frac{e^{-x}}{x} + \int_1^x \frac{e^{-t}}{t} dt = e^{-x} + \int_1^x \frac{e^{-t}}{t} dt$$

$$y'' = -e^{-x} + \frac{e^{-x}}{x}$$

$$\begin{aligned} x^2 y'' + (x^2 - x) y' + (1 - x) y &= (-x^2 e^{-x} + x e^{-x}) \\ &\quad + \left(x^2 e^{-x} + x^2 \int_1^x \frac{e^{-t}}{t} dt - x e^{-x} - x \int_1^x \frac{e^{-t}}{t} dt \right) \\ &\quad + \left(x \int_1^x \frac{e^{-t}}{t} dt - x^2 \int_1^x \frac{e^{-t}}{t} dt \right) = 0 \end{aligned}$$

$$\begin{aligned}
30. \quad y &= \sin x \int_0^x e^{t^2} \cos t \, dt - \cos x \int_0^x e^{t^2} \sin t \, dt \\
y' &= \sin x \left(e^{x^2} \cos x \right) + \cos x \int_0^x e^{t^2} \cos t \, dt - \cos x \left(e^{x^2} \sin x \right) + \sin x \int_0^x e^{t^2} \sin t \, dt \\
&= \cos x \int_0^x e^{t^2} \cos t \, dt + \sin x \int_0^x e^{t^2} \sin t \, dt \\
y'' &= \cos x \left(e^{x^2} \cos x \right) - \sin x \int_0^x e^{t^2} \cos t \, dt + \sin x \left(e^{x^2} \sin x \right) + \cos x \int_0^x e^{t^2} \sin t \, dt \\
&= e^{x^2} (\cos^2 x + \sin^2 x) - \left(\overbrace{\sin x \int_0^x e^{t^2} \cos t \, dt - \cos x \int_0^x e^{t^2} \sin t \, dt}^y \right) \\
&= e^{x^2} - y
\end{aligned}$$

$$y'' + y = e^{x^2} - y + y = e^{x^2}$$

31. Using implicit differentiation we get

$$\begin{aligned}
x^3 y^3 &= x^3 + 5 \\
3x^2 \cdot y^3 + x^3 \cdot 3y^2 \frac{dy}{dx} &= 3x^2 \\
\frac{3x^2 y^3}{3x^2 y^2} + \frac{x^3 3y^2}{3x^2 y^2} \frac{dy}{dx} &= \frac{3x^2}{3x^2 y^2} \\
y + x \frac{dy}{dx} &= \frac{1}{y^2}
\end{aligned}$$

32. Using implicit differentiation we get

$$\begin{aligned}
(x-7)^2 + y^2 &= 1 \\
2(x-7) + 2y \frac{dy}{dx} &= 0 \\
2y \frac{dy}{dx} &= -2(x-7) \\
y \frac{dy}{dx} &= -(x-7) \\
\left(y \frac{dy}{dx} \right)^2 &= (x-7)^2 \\
\left(\frac{dy}{dx} \right)^2 &= \frac{(x-7)^2}{y^2}
\end{aligned}$$

Now from the original equation, isolating the first term leads to $(x-7)^2 = 1-y^2$. Continuing from the last line of our proof we now have

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x-7)^2}{y^2} = \frac{1-y^2}{y^2} = \frac{1}{y^2} - 1$$

Adding 1 to both sides leads to the desired result.

33. Using implicit differentiation we get

$$y^3 + 3y = 2 - 3x$$

$$3y^2y' + 3y' = -3$$

$$y^2y' + y' = -1$$

$$(y^2 + 1)y' = -1$$

$$y' = \frac{-1}{y^2 + 1}$$

Differentiating the last line and remembering to use the quotient rule on the right side leads to

$$y'' = \frac{2yy'}{(y^2 + 1)^2}$$

Now since $y' = -1/(y^2 + 1)$ we can write the last equation as

$$y'' = \frac{2y}{(y^2 + 1)^2}y' = \frac{2y}{(y^2 + 1)^2} \frac{-1}{(y^2 + 1)} = 2y \left(\frac{-1}{y^2 + 1} \right)^3 = 2y(y')^3$$

which is what we wanted to show.

34. Using implicit differentiation we get

$$y = e^{-xy}$$

$$y' = e^{-xy}(-y - xy')$$

$$ye^{-xy} + xe^{-xy}y' + y' = 0$$

$$ye^{-xy} + (xe^{-xy} + 1)y' = 0$$

Now since $y = e^{-xy}$, substitute this into the last line to get

$$yy + (xy + 1)y' = 0$$

or $(1 + xy)y' + y^2 = 0$ which is what we wanted to show.

- 35.** Substituting $y = c_1 + \cos 3x$, $y' = -3 \sin 3x$, and $y'' = -9 \cos 3x$ into the left-hand side of the differential equation gives

$$y'' + 9y = -9 \cos 3x + 9(c_1 + \cos 3x) = -9 \cos 3x + 9c_1 + 9 \cos 3x = 9c_1.$$

Setting the result equal to the right-hand side of the differential equation yields $9c_1 = 5$ or $c_1 = \frac{5}{9}$. Thus $y = \frac{5}{9} + \cos 3x$.

- 36.** Substituting $y = c_1 + c_2x$ and $y' = c_2$ into the left-hand side of the differential equation gives

$$y' + 2y = c_2 + 2(c_1 + c_2x) = c_2 + 2c_1 + 2c_2x.$$

Setting the result equal to the right-hand side of the differential equation yields

$$c_2 + 2c_1 + 2c_2x = 3x$$

$$(c_2 + 2c_1) + 2c_2x = 0 + 3x.$$

Therefore, $2c_2 = 3$ or $c_2 = \frac{3}{2}$, and $\frac{3}{2} + 2c_1 = 0$ or $c_1 = -\frac{3}{2} \cdot \frac{1}{2} = -\frac{3}{4}$. Thus $y = -\frac{3}{4} + \frac{3}{2}x$.

- 37.** Differentiating and isolating c gives

$$y' = -ce^{-x} + 4$$

$$ce^{-x} = 4 - y'$$

$$c = e^x (4 - y').$$

Substituting the last equation into $y = ce^{-x} + 4x - 6$ and simplifying yields

$$y = e^x (4 - y') e^{-x} + 4x - 6$$

$$y = 4 - y' + 4x - 6$$

$$y' + y = 4x - 2.$$

Alternatively, isolating c in $y = ce^{-x} + 4x - 6$ gives

$$ce^{-x} = y - 4x + 6$$

$$c = e^x (y - 4x + 6).$$

Differentiating the last equation yields

$$0 = e^x (y - 4x + 6) + e^x (y' - 4)$$

$$0 = e^x (y' + y - 4x + 2)$$

$$0 = y' + y - 4x + 2$$

$$y' + y = 4x - 2.$$

38. The slope $\frac{dy}{dx} = 6 - 2x$. Integrating with respect to x we have $y = 6x - x^2 + c_1$. Since the graph must pass through the origin $(0, 0)$, we have

$$y(0) = 6(0) - (0)^2 + c_1$$

$$0 = c_1.$$

Thus $y = 6x - x^2$.

In Problems 39–42, $y = c_1e^{-3x} + c_2e^x + 4x$ is given as a two-parameter family of solutions of the second-order differential equation $y'' + 2y' - 3y = -12x + 8$.

39. If $y(0) = 0$ and $y'(0) = 0$, then

$$c_1 + c_2 = 0$$

$$-3c_1 + c_2 = -4$$

Subtracting the second equation from the first gives us $4c_1 = 4$ or $c_1 = 1$, and thus $c_2 = -1$. Therefore $y = e^{-3x} - e^x + 4x$.

40. If $y(0) = 5$ and $y'(0) = -11$, then

$$c_1 + c_2 = 5$$

$$-3c_1 + c_2 = -15$$

Subtracting the second equation from the first gives us $4c_1 = 20$ or $c_1 = 5$, and thus $c_2 = 0$. Therefore $y = 5e^{-3x} + 4x$.

41. If $y(1) = -2$ and $y'(1) = 4$, then

$$c_1e^{-3} + c_2e = -6$$

$$-3c_1e^{-3} + c_2e = 0$$

Subtracting the second equation from the first gives us $4c_1 = -6$ or $c_1 = -\frac{3}{2}e^3$, and thus $c_2 = -\frac{9}{2}e^{-1}$. Therefore $y = -\frac{3}{2}e^{-3x+3} - \frac{9}{2}e^{x-1} + 4x$.

42. If $y(-1) = 1$ and $y'(-1) = 1$, then

$$c_1e^3 + c_2e^{-1} = 5$$

$$-3c_1e^3 + c_2e^{-1} = -3$$

Subtracting the second equation from the first gives us $4c_1 = 8$ or $c_1 = 2e^{-3}$, and thus $c_2 = 3e$. Therefore $y = 2e^{-3x-3} + 3e^{x+1} + 4x$.

43. We are to use the Leibniz's rule

$$\frac{d}{dx} \int_{u(x)}^{v(x)} F(x, t) dt = F(x, v(x)) \frac{dv}{dx} - F(x, u(x)) \frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} F(x, t) dt$$

Since $y(x) = \frac{1}{3} \int_0^x f(t) \sin(3x - 3t) dt$, take $F(x, t) = f(t) \sin(3x - 3t)$, $u(x) = 0$, and $v(x) = x$ to get

$$\begin{aligned} \frac{d}{dx} y(x) &= \frac{d}{dx} \frac{1}{3} \int_0^x f(t) \sin(3x - 3t) dt \\ &= \frac{1}{3} \left[F(x, x) \cdot 1 - F(x, 0) \cdot 0 + \int_0^x \frac{\partial}{\partial x} f(t) \sin(3x - 3t) dt \right] \\ &= \frac{1}{3} \left[f(x) \sin(3x - 3x) + \int_0^x 3f(t) \cos(3x - 3t) dt \right] \\ &= \int_0^x f(t) \cos(3x - 3t) dt \end{aligned}$$

Apply the Leibniz's rule a second time to $y' = \int_0^x f(t) \cos(3x - 3t) dt$ by taking $F(x, t) = f(t) \cos(3x - 3t)$, $u(x) = 0$, and $v(x) = x$ to get

$$\begin{aligned} \frac{d}{dx} y'(x) &= \frac{d}{dx} \int_0^x f(t) \cos(3x - 3t) dt \\ &= F(x, x) \cdot 1 - F(x, 0) \cdot 0 + \int_0^x \frac{\partial}{\partial x} f(t) \cos(3x - 3t) dt \\ &= f(x) \cos(3x - 3x) + \int_0^x -3f(t) \sin(3x - 3t) dt \\ &= f(x) - 3 \int_0^x f(t) \sin(3x - 3t) dt \end{aligned}$$

Therefore $y'' = f(x) - 3 \int_0^x f(t) \sin(3x - 3t) dt$. Now substituting y and y'' into the differential equation we get

$$y'' + 9y = f(x) - 3 \int_0^x f(t) \sin(3x - 3t) dt + 9 \cdot \frac{1}{3} \int_0^x f(t) \sin(3x - 3t) dt = f(x)$$

44. We are to use the Leibniz's rule

$$\frac{d}{dx} \int_{u(x)}^{v(x)} F(x, t) dt = F(x, v(x)) \frac{dv}{dx} - F(x, u(x)) \frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} F(x, t) dt$$

Since $y(x) = \int_0^\pi e^{x \cos t} dt$, take $F(x, t) = e^{x \cos t}$, $u(x) = 0$, and $v(x) = \pi$ to get

$$\begin{aligned} \frac{d}{dx} y(x) &= \frac{d}{dx} \int_0^\pi e^{x \cos t} dt \\ &= F(x, \pi) \cdot 0 - F(x, 0) \cdot 0 + \int_0^\pi \frac{\partial}{\partial x} e^{x \cos t} dt \\ &= \int_0^\pi \cos t e^{x \cos t} dt \end{aligned}$$

Applying the Leibniz's rule a second time to $y'(x) = \int_0^\pi \cos t e^{x \cos t} dt$ by taking $F(x, t) = \cos t e^{x \cos t}$, $u(x) = 0$, and $v(x) = \pi$ to get

$$\begin{aligned} \frac{d}{dx} y'(x) &= \int_0^\pi \cos t e^{x \cos t} dt \\ &= F(x, \pi) \cdot 0 - F(x, 0) \cdot 0 + \int_0^\pi \frac{\partial}{\partial x} \cos t e^{x \cos t} dt \\ &= \int_0^\pi \cos^2 t e^{x \cos t} dt \end{aligned}$$

An alternative form of y' can be obtained by integrating by parts with respect to t :

$$y' = \sin t \cdot e^{x \cos t} \Big|_0^\pi + \int_0^\pi x e^{\cos t} \sin^2 t dt = \int_0^\pi x e^{x \cos t} \sin^2 t dt = x \int_0^\pi e^{x \cos t} \sin^2 t dt.$$

We use this last form of y' instead of the first in the differential equation:

$$\begin{aligned} xy'' + y' &= x \int_0^\pi e^{x \cos t} \cos^2 t dt + x \int_0^\pi e^{x \cos t} \sin^2 t dt \\ &= x \int_0^\pi \left(\overbrace{\cos^2 t + \sin^2 t}^1 \right) e^{x \cos t} dt = x \int_0^\pi \overbrace{e^{x \cos t}}^y dt = xy \\ xy'' + y' &= xy \end{aligned}$$

Therefore, $xy'' + y' - xy = 0$.

45. From the graph we see that estimates for y_0 and y_1 are $y_0 = -3$ and $y_1 = 0$.

46. The differential equation is

$$\frac{dh}{dt} = -\frac{cA_0}{A_w} \sqrt{2gh}.$$

Using $A_0 = \pi(1/24)^2 = \pi/576$, $A_w = \pi(2)^2 = 4\pi$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/576}{4\pi} \sqrt{64h} = -\frac{c}{288} \sqrt{h}.$$

47. Let $x(t)$ represent the height of the top of the rope at any time t with the positive direction upward as indicated. The weight of the portion of the rope off the ground is given by $W = (x \text{ ft}) \cdot (1 \text{ lb/ft}) = x$. The mass of the rope is $m = W/g = x/32$. The net force is $F = 5 - W = 5 - x$. Now by Newton's second law we get

$$F = \frac{d}{dt}(mv) = \frac{d}{dt} \left(\frac{x}{32} \cdot v \right) = 5 - x$$

Now expand the derivative and remember that $v = dx/dt$ to get

$$\frac{d}{dt} \left(\frac{x}{32} \cdot v \right) = 5 - x$$

$$\frac{1}{32} \left(x \frac{dv}{dt} + v \frac{dx}{dt} \right) = 5 - x$$

$$x \frac{d}{dt} \left(\frac{dx}{dt} \right) + \left(\frac{dx}{dt} \right) \frac{dx}{dt} = 160 - 32x$$

$$x \frac{d^2x}{dt^2} + \left(\frac{dx}{dt} \right)^2 + 32x = 160$$

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TEST BANK

Advanced Engineering Mathematics, Seventh Edition

Dennis G. Zill

Chapter 1 Test Bank

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Chapter: Chapter 01 – Test Bank

Multiple Choice

1. Classify the following differential equation by order and linearity. $\frac{d^3y}{dx^3} = \left(\frac{d^2y}{dx^2}\right)^4$

A) 3rd order nonlinear

B) 8th order nonlinear

C) 4th order linear

D) 3rd order linear

Ans: A

Complexity: Easy

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

2. Classify the following differential equation by order and linearity. $y'' = \sqrt{5x^6 + (\sin x)y^{(8)}}$

A) 8th order nonlinear

B) 4th order linear

C) 2nd order linear

D) 2nd order nonlinear

Ans: A

Complexity: Easy

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

3. Classify the following differential equation by order and linearity. $(1 - e^y)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = \cos 2t$

A) 2nd order linear

B) 2nd order nonlinear

C) 1st order linear

D) 1st order nonlinear

Ans: A

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Chapter 1 Test Bank

Complexity: Easy

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

4. Classify the following differential equation by order and linearity.

$$\ddot{x} - (1 - \dot{x}^3)\dot{x} + x = 2$$

A) 2nd order nonlinear

B) 2nd order linear

C) 1st order linear

D) 1st order nonlinear

Ans: A

Complexity: Easy

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

5. For the differential equation $y' = \frac{\tan(y)}{x}$, the Existence-Uniqueness Theorem (Theorem 1.2.1) guarantees the existence of a unique solution whose graph passes through:

A) $(\frac{\pi}{2}, \pi)$.

B) $(1, \frac{\pi}{2})$.

C) $(\pi, \frac{3\pi}{2})$.

D) $(0, \pi)$.

Ans: A

Complexity: Moderate

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

6. For the DE $y' = \frac{2y^2+5}{\ln(4y-3x)}$, the Existence-Uniqueness Theorem (Theorem 1.2.1) guarantees the existence of a unique solution whose graph passes through:

A) $(\frac{2}{3}, \frac{3}{2})$.

B) $(0, \frac{1}{4})$.

C) $(1, \frac{3}{4})$.

D) $(-7, -5)$.

Ans: A

Complexity: Moderate

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

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7. For the DE $\frac{dy}{dx} = |\sin y| \sin^{-1}(x)$, the Existence-Uniqueness Theorem (Theorem 1.2.1) guarantees the existence of a unique solution whose graph passes through:

A) $(\frac{1}{2}, \frac{\pi}{4})$.

B) $(\frac{\pi}{2}, \frac{\pi}{3})$.

C) $(0, 0)$.

D) $(\frac{2}{3}, \pi)$.

Ans: A

Complexity: Moderate

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

8. $y = -x \ln |x| + cx$ is a one-parameter family of solutions to the DE $x \frac{dy}{dx} = y - x$. Which of the following is the largest interval of existence for a solution to the IVP $x \frac{dy}{dx} = y - x$, $y(2) = -7$?

A) $(0, \infty)$

B) $(-\infty, 0)$

C) $(-7, 2)$

D) $(2, \infty)$

Ans: A

Complexity: Easy

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

9. $y = \frac{1}{4}e^{3x} + ce^{-x}$ is a one-parameter family of solutions to the DE $\frac{dy}{dx} + y = e^{3x}$. Which of the following is the largest interval of existence for a solution to the IVP $\frac{dy}{dx} + y = e^{3x}$, $y(-4) = 15$?

A) $(-\infty, \infty)$

B) $(-\infty, 0)$

C) $(-4, 15)$

D) $(-4, \infty)$

Ans: A

Complexity: Easy

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

10. $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the initial conditions $x(\pi/3) =$

$\frac{1}{2}$, $x'(\pi/3) = -\frac{\sqrt{3}}{2}$.

A) $x = \cos t$

B) $x = -\sin t$

C) $x = \cos t + \sin t$

D) $x = -\cos t + \frac{1}{2}\sin t$

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Ans: A

Complexity: Moderate

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

11. A hot cup of coffee is initially at 180°F , sits in a room with a constant air temperature of 72°F , and cools according to Newton's law of cooling. Write an initial-value problem describing the temperature of the coffee at time t , $T(t)$.

A) $\frac{dT}{dt} = k(T - 72), \quad T(0) = 180$

B) $\frac{dT}{dt} = k(T - 80), \quad T(0) = 72$

C) $\frac{dT}{dt} = 72(T - k), \quad T(0) = 180$

D) $\frac{dT}{dt} = 180(72 - T), \quad T(0) = k$

Ans: A

Complexity: Easy

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

12. Suppose that the birth rate of a certain population is proportional to the current size of the population with proportionality constant b . Suppose that the death rate is also proportional to the current size of the population with proportionality constant d . If the change in size of the population depends solely on the births and deaths that occur, which of the following could represent the size of the population P as a function of time t ?

A) $P(t) = e^{(b-d)t + \ln(4)}$

B) $P(t) = 6e^{(d-b)t}$

C) $P(t) = e^{bt} - e^{dt}$

D) $P(t) = 6e^{(b+d)t}$

Ans: A

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

Title: Introduction to Differential Equations

13. Suppose two particles A and B are moving along the x -axis. At time t , particle A is located at the point on the x -axis with coordinate $5 + 7t$. When $t = 0$, B is at the origin. B 's velocity is along the positive x -axis and is proportional to the distance between A and B with proportionality constant $-2k$. The initial value problem that models B 's position x on the axis as a function of t is:

A) $\frac{dx}{dt} = k(5 + 7t - x), \quad x(0) = 0.$

B) $\frac{dx}{dt} = k(x + 5 - 7t), \quad x(0) = 0.$

C) $\frac{dx}{dt} = 7, \quad x(0) = 5.$

D) $\frac{dx}{dt} = k(5 - x), \quad x(0) = 0.$

Ans: A

Complexity: Moderate

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Ahead: Differential Equations as Mathematical Models

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Title: Introduction to Differential Equations

True/False

1. True or False? The following differential equation is linear: $\frac{1}{x}y''' + (3 - x^2)(y'' - y') + e^xy = 3y$.

Ans: True

Complexity: Easy

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

2. True or False? The following differential equation is linear: $\frac{d^3u}{dt^3} + 4^u = \sin t$.

Ans: False

Complexity: Easy

Ahead: Definitions and Terminology

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3. True or False? There is a constant solution to the differential equation: $2y'' - 3y' = y^2 - 6y + 10$.

Ans: False

Complexity: Easy

Ahead: Definitions and Terminology

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Title: Introduction to Differential Equations

4. True or False? There is a constant solution to the differential equation: $(y^2 - 5)y'' = 2 - y'$.

Ans: False

Complexity: Easy

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

5. True or False? $y = 2e^{-x} + 3e^{4x}$ is a solution to the differential equation $y'' - 3y' - 4y = 0$.

Ans: True

Complexity: Easy

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6. True or False? An implicit solution to a differential equation is a relation that defines one or more solutions to the differential equation.

Ans: True

Complexity: Easy

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

7. True or False? A singular solution to a differential equation is a solution that cannot be obtained by specializing any of the parameters in the family of solutions.

Ans: True

Complexity: Easy

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

8. True or False? The function $y = \frac{e^t}{1+e^t}$ is a solution to the IVP $y' - y + y^2 = 0$, $y(0) = \frac{1}{2}$.

Ans: True

Complexity: Moderate

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

9. True or False? The function $y = -x - 1 + \tan(x + 1)$ is a solution to the IVP $\frac{dy}{dx} = (x + y + 1)^2$, $y(-1) = 0$.

Ans: True

Complexity: Moderate

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

10. True or False? $y = c_1x^{-1} + c_2x - \ln x$ is a two-parameter family of solutions to the differential equation $x^2y'' + xy' - y = \ln x$.

Ans: True

Complexity: Easy

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

11. True or False? $y = \frac{x^3}{7} - \frac{x}{5} + \frac{72}{35}x^{-4}$ is a solution to the IVP $xy' + 4y = x^3 - x$, $y(1) = 2$.

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Ans: True

Complexity: Moderate

Ahead: Initial-Value Problems

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Title: Introduction to Differential Equations

12. True or False? Every initial-value problem has a unique solution.

Ans: False

Complexity: Easy

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

13. True or False? A solution to an initial-value problem is always defined on $(-\infty, \infty)$.

Ans: False

Complexity: Easy

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

14. True or False? Consider a population where the number of births and deaths are proportional to the size of the population with proportionality constants b and d respectively. Suppose also that the population experiences a constant net immigration rate of r . Then the differential equation that models the population P as a function of time t is $\frac{dP}{dt} = (b - d + r)P$.

Ans: False

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

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Essay

1. Find values of m so that the function $y = e^{mx}$ is a solution of the differential equation $y'' + 2y' - 35y = 0$.

Ans: $m = -7, 5$

Complexity: Easy

Ahead: Definitions and Terminology

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2. Find values of m so that the function $y = x^m$ is a solution of the following differential equation.

$$xy'' + 5xy' - 12y = 0$$

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Ans: $m = -6, 2$

Complexity: Moderate

Ahead: Definitions and Terminology

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3. Classify the following differential equation by order and linearity.

$$\frac{d^2 y}{dx^2} = \left(\frac{dy}{dx}\right)^5 - \frac{d^4 y}{dx^4} + \left(\frac{d^3 y}{dx^3}\right)^2$$

Ans: 4th order nonlinear

Complexity: Easy

Ahead: Definitions and Terminology

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4. Describe how to identify the order of a differential equation.

Ans: The order of a differential equation is the order of the highest derivative in the equation.

Complexity: Easy

Ahead: Definitions and Terminology

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5. Give an example of a linear differential equation and an example of a nonlinear differential equation. Clearly indicate which is which.

Ans: Answers will vary.

Complexity: Difficult

Ahead: Definitions and Terminology

Subject: Chapter 1

Title: Introduction to Differential Equations

6. $y = c_1 e^{-4x} + c_2 e^{2x}$ is a two-parameter family of solutions of the second-order differential equation $y'' + 2y' - 8y = 0$, $y(0) = 0$, $y'(0) = 12$. Find a solution of the second-order IVP $y'' + 2y' - 8y = 0$, $y(0) = 0$, $y'(0) = 12$.

Ans: $y = -2e^{-4x} + 2e^{2x}$

Complexity: Moderate

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

7. $y = c_1 \cos 3t + c_2 \sin 3t$ is a two-parameter family of solutions of the second-order differential equation $y'' + 9y = 0$. Find a solution of the second-order differential equation

$y'' + 9y = 0$, $y(0) = 1$, $y'\left(\frac{\pi}{3}\right) = 2$.

Ans: $y = \cos 3t - \frac{2}{3} \sin 3t$

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Complexity: Moderate

Ahead: Initial-Value Problems

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8. $y = x^{-1} \ln x + cx^{-1}$ is a family of solutions to the differential equation $x^2 y' + xy = 1$. Find a solution to the IVP $x^2 y' + xy = 1$, $y(1) = 9$.

Ans: $y = x^{-1} \ln x + 9x^{-1}$

Complexity: Moderate

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

9. A certain island initially contains 500 inhabitants who communicate only by word of mouth. A boat containing 10 people from another country arrives on the island with news of the outside world. Determine a differential equation for the number of people $x(t)$ who have heard the news at time t if the rate at which the news spreads is proportional to the number of interactions between those who have heard the news and those who have not. Treating this as an initial-value problem, what is the initial condition? (Assume throughout the problem that the 10 outsiders stay indefinitely and become part of the islander's population.)

Ans: $\frac{dx}{dt} = kx(510 - x)$, $x(0) = 10$

Complexity: Difficult

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

10. Suppose a cylindrical tank with radius 3 meters and height 9 meters is resting on its circular base. Water is leaking from the tank through a circular hole of area A at its bottom. The volume of water leaving the tank per second is equal to $cA\sqrt{2gh}$, where h is the height of the water in the tank, g is the acceleration due to gravity, and c is an empirical constant with $0 < c < 1$. If the tank is full at time $t = 0$ and the radius of the hole is 0.01 meters, determine a differential equation and initial condition for the height h of water at time t .

Ans: $\frac{dh}{dt} = -c \left(\frac{01}{3}\right)^2 \sqrt{2gh}$, $h(0) = 9$

Complexity: Difficult

Ahead: Initial-Value Problems

Subject: Chapter 1

Title: Introduction to Differential Equations

11. Find the value of w , which makes $y = c_1 \sin w t + c_2 \cos w t$ a two-parameter family of solutions of the differential equation $y'' + 18y = 0$.

Ans: $w = 3\sqrt{2}$

Complexity: Moderate

Ahead: Initial-Value Problems

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Title: Introduction to Differential Equations

12. Suppose that a large mixing tank initially holds 500 gallons of water in which 75 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 6 gal/min, and when the solution is well stirred, it is pumped out at the same rate. Determine a differential equation for the amount $A(t)$ of salt in the tank at time t . What is $A(0)$?

Ans: $\frac{dA}{dt} = \frac{-3A}{250}$, $A(0) = 75$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

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13. Suppose that a large mixing tank initially holds 500 gallons of water in which 75 pounds of salt have been dissolved. Water containing a salt concentration of $\frac{1}{2}(1 + 3 \sin t)$ lb/gal flows into the tank at a rate of 2 gal/min, and when the solution is well stirred, it is pumped out at the same rate. Determine a differential equation for the amount $A(t)$ of salt in the tank at time t . What is $A(0)$?

Ans: $\frac{dA}{dt} = 1 + \sin t - \frac{A}{250}$, $A(0) = 75$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

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14. Suppose that a large mixing tank initially holds 400 gallons of water in which 65 pounds of salt have been dissolved. Another brine solution is pumped into the tank at a rate of 6 gal/min, and when the solution is well stirred, it is pumped out at a slower rate of 5 gal/min. If the concentration of the solution entering is 3 lb/gal, determine a differential equation for the amount $A(t)$ of salt in the tank at time t . What is $A(0)$?

Ans: $\frac{dA}{dt} = 18 - \frac{5A}{400+t}$, $A(0) = 65$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

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15. Suppose that a large mixing tank initially holds 400 gallons of water in which 65 pounds of salt have been dissolved. Another brine solution is pumped into the tank at a rate of 6 gal/min, and when the solution is well stirred, it is pumped out at a faster rate of 8 gal/min. If the concentration of the solution entering is 3 lb/gal, determine a differential equation for the amount $A(t)$ of salt in the tank at time t .

Ans: $\frac{dA}{dt} = 18 - \frac{8A}{400-2t}$, $A(0) = 65$

Complexity: Moderate

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16. Determine a differential equation for the velocity $v(t)$ of a falling body of mass m if air resistance is proportional to the square root of the cube of the instantaneous velocity.

Ans: $m \frac{dv}{dt} = mg - kv^{3/2}$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

Title: Introduction to Differential Equations

17. Suppose water is leaking from a spherical tank with radius 5 meters through a circular hole of area A at its bottom. The volume of water leaving the tank per second is equal to $cA\sqrt{2gh}$, where h is the height of the water in the tank, g is the acceleration due to gravity, and c is an empirical constant with $0 < c < 1$. If the tank is half-full at time $t = 0$ and the radius of the hole is 0.02 meters, determine a differential equation and initial condition for the height h of water at time t .

Ans: $\frac{dh}{dt} = -c \left(\frac{.02}{10h-h^2} \right)^2 \sqrt{2gh}, \quad h(0) = 5$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

Title: Introduction to Differential Equations

18. Suppose the temperature of a room fluctuates with time so that the temperature at time t is given by $R(t) = 79 + 11 \cos t$. A 200° F cup of hot tea is placed on a table in the room at time $t = 12$. Determine a differential equation for the temperature of the tea $T(t)$ if the magnitude of the constant of proportionality is $|k| = 0.3$.

Ans: $\frac{dT}{dt} = -0.3(T - 79 - 11 \cos t), \quad T(12) = 200$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

Title: Introduction to Differential Equations

19. Suppose a dog is chasing a rabbit along the y -axis, and the coordinate of the rabbit's position at time t is given by $\frac{1}{3}t^2 + 4t$. If the dog's speed is equal to one half the distance between the two animals, and the dog's coordinate at $t = 0$ is $y = -4$, determine a differential equation for the position of the dog.

Ans: $\frac{dD}{dt} = \frac{1}{2} \left(\frac{1}{3}t^2 + 4t - D \right), \quad D(0) = -4$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

Title: Introduction to Differential Equations

20. Consider a population where the number of births and deaths are proportional to the size of the population with proportionality constants b and d respectively. Suppose also that people immigrate to the country at a constant rate of r . If people emigrate from the country at a rate proportional to the population P with proportionality constant w , determine a differential equation that models P as a function of time t .

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Ans: $\frac{dP}{dt} = (b - d - w)P + r$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

Title: Introduction to Differential Equations

21. Water drains out of a tank according to Torricelli's law. If the upper surface of the water is always 5 times the area of the drain hole and the acceleration due to gravity is 32, then write a differential equation describing $h(t)$, the height of the water in the tank at time t .

Ans: $\frac{dh}{dt} = -\frac{1}{5}\sqrt{2 \cdot 32 \cdot h}$

Complexity: Moderate

Ahead: Differential Equations as Mathematical Models

Subject: Chapter 1

Title: Introduction to Differential Equations